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A CLASSIFICATION OF QUADRATIC VECTORS.

By Frank Lauren Hitchcock.

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A CLASSIFICATION OF QUADRATIC VECTORS.

By Frank Lauren Hitchcock.

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INTRODUCTION.

In this paper it is shown that three ternary quadratic forms X, Y, and Z, homogeneous polynomials of the second degree in the variables x, y, and z, can in general be thrown into the form

$$X = vw_1 - v_1w + tx, \quad Y = wu_1 - w_1u + ty, \quad Z = uv_1 - u_1v + tz, \quad \text{(I)}$$

where the seven letters: u, v, w; u_1, v_1, w_1; and t; — denote linear forms, that is, they are homogeneous polynomials of the first degree in x, y, z.

In the language of vector algebra, the scalars X, Y, Z are the components of a vector $F(\rho)$, or simply $F\rho$. The linear forms u, v, and w are components of a linear vector $\phi\rho$, and u_1, v_1, w_1 are components of a second linear vector $\theta\rho$. The linear form t is regarded as the scalar product of a constant vector δ and the point-vector ρ. The above statement translates into the vector equation

$$F\rho = V\phi\rho\,\theta\rho + \rho S\delta\rho, \qquad \text{(II)}$$

or, in words, a quadratic vector function, homogeneous in ρ, can be expressed as the vector product of two linear vector functions, aside from a properly chosen scalar multiple of the point-vector.

The significance of this result lies, in part, in the fact that, in various problems depending on three quantics, the term in ρ may be taken arbitrarily. For example, the differential equation

$$(yZ - zY)dx + (zX - xZ)dy + (xY - yX)\,dz = 0$$

in vector language becomes $S\rho F\rho d\rho = 0$, or $Sd\rho V\rho F\rho = 0$, and is thus independent of the term in ρ and of the linear form t. The vector $V\rho F\rho$ may in general be factored into $V\rho V\phi\rho\theta\rho$.

It is also shown that the vector δ, equivalent to the linear form t, may in general be determined in thirty-five ways. The three scalar equations (I), equivalent to the vector equation (II), (X, Y, and Z being given and the right members to be determined), are in general equivalent to eighteen quadratic equations. The solution depends upon an equation of the seventh degree, which determines seven sets

of values of x, y, and z, called axes of the quadratic vector. When these axes are found we can then find δ and t.

The thirty-five ways of finding t are diminished in number by various special relations among the constituents of X, Y, and Z. These restrictions are simply expressible in terms of configurations of the axes. It is shown that at least one value of t can be found, that is, that the eighteen quadratics have a solution, except in two cases.

Methods of finding δ or t are worked out for all possible configurations of the axes. This is done by means of normal forms, or model vectors, including various classes of quadratic vectors.

The use made of the technical methods of vector algebra is slight in the first part of the paper, more extended when dealing with conditions of multiplicity among the axes. The results, while they were invariably obtained by vector algebra, can be verified in most instances by the reader who has only a slight acquaintance with these methods.

The classification of vectors under normal or type forms is worked out on the following scheme: —

Class I. Seven distinct axes. Type form (31).

Class II. At least one axis is of order two or more. These are subdivided into

1. Vectors having one or more double axes but no triple axis. The double axes may be one, two, or three, in number. Types Art. 16.

2. Vectors having at least one triple axis but no quadruple axis. Of these, there may be either one, or two, double axes. We may also have two distinct triple axes. Types Art. 25.

3. Vectors having an axis of order four or higher. These are of two distinct kinds, according as the multiple axis is —

Class 1°. A double element of all cones $V\rho F\rho = 0$. Type forms are discussed in Art. 27 and 37.

Class 2°. Not a double element for all cones $V\rho F\rho = 0$. Types Art. 30, including the two forms (185) and (195).

Class III. The number of axes is infinite. There are three special forms (57). The vector $V\rho F\rho$ possesses a scalar factor, and is said to be reducible. The equations (II) are not considered with reference to this class, the forms (57) being equally simple.

PART ONE.

1. In many problems of Geometry and of Physics we meet with quantities which, occurring at the points of a portion of space, possess a definite direction, as well as magnitude or length along that direction. The velocity of a fluid at a point, and the force at a point due to the attraction of an assigned distribution of matter, are familiar examples. If such a directed quantity, or geometrical vector, be resolved along chosen axes of reference, it yields components X, Y, and Z, which are ordinary, or scalar, functions of x, y, and z, the coördinates of a point in space. Approaching the matter from the side of Algebra, both the independent variables and the component functions will, in general, be free to take on complex values.

If we agree upon the following four conventions,—

1. The vectors whose components are (1, 0, 0,), (0, 1, 0), and (0, 0, 1) are denoted, respectively, by i, j, and k.
2. Multiplication of a vector by a scalar means multiplication of all components of the vector by the scalar.
3. Vectors are added by adding their corresponding components.
4. Equality of vectors implies equality of corresponding components,—

it follows that we may write, for the vector ρ from the origin to a point in space,

$$\rho = ix + jy + kz \qquad (1).$$

A vector function of ρ may be denoted by $F(\rho)$, whence

$$F(\rho) = iX + jY + kZ \qquad (2).$$

My present object is to contribute something toward a theory of those vectors whose components are homogeneous polynomials of the second degree in the variables x, y, z. That is, X, Y, and Z are ternary quadratic forms. The theory of my vectors will then be, up to a certain point, in close relation with the theory of a set of three forms, but the name vector implies also a definite order, X, Y, Z, among these forms; and we have, moreover, the above defined process of addition of vectors.

2. The classification which I propose to make of various types of quadratic vector depends upon the existence of directions called *axes*, which are directions of the point-vector ρ such that we have simultaneously

$$yZ - zY = 0, \quad zX - xZ = 0, \quad xY - yX = 0 \tag{3}$$

Geometrically stated, an axis of $F(\rho)$ is a direction of ρ such that $F(\rho)$ either vanishes in all its components or is parallel to ρ. Those axes, if any, for which $F(\rho)$ vanishes, may be called the *zeros* of $F(\rho)$.

It is well known that the number of sets of values of the ratios of x, y, and z satisfying equations (3) is, when X, Y, and Z are ternary forms of degree p, in general equal to $p^2 + p + 1$.[1] A quadratic vector form has therefore, in general, seven axes.

Of these seven axes, however, some may be in coincidence, giving multiple axes.

If, on the other hand, there are more than seven axes, there are an infinite number. For let there be eight or more axes, and suppose their number finite. Equations (3), if satisfied, will subsist after a change of the coördinate system. We may therefore suppose no axis to lie in the plane $z = 0$. The first two of equations (3) will have ten or more solutions, viz. the two lines of intersection of the plane $z = 0$ with the quadric cone $Z = 0$, and the eight or more axes of the vector. But two cubic equations with ten solutions have an infinite number. The third equation is a consequence of the first two wherever z is not zero. Hence the three equations have a common linear or quadratic factor, and the number of axes cannot be finite.

These facts suggest a division of quadratic vectors into three types,—

I. General type. There are seven, and only seven distinct axes.

---

1 For two proofs from very different points of view, see Darboux, "Memoire sur les Equations Differentieles Algébriques," Bul. Sci. Math. 13, (1878), p. 83, where the solutions of (3) give the singular points of a differential equation; and Clebsch, "Lecons sur la Geometrie," (Tr. Benoist), t. II p. 113 and t. III p. 435, (Vorl. u. Geom. Bd. I s. 390, 1001), where these equations are connected with a quadratic complex.

Again, if we let a point transformation in homogeneous coördinates be defined by the equations $x = X$, $y = Y$, $z = Z$, then (3) gives the fixed points, together with the singular points.

A short proof for the present case is as follows,— let the first two equations of (3) define two cubic curves. They intersect, in general, in nine points, of which two are the intersections of the line $z = 0$ and the conic $Z = 0$. If z and Z are not both zero the determinant $xY - yX$ must vanish. Hence at 9–2 points equations (3) hold simultaneously.

II. The number of distinct axes is less than seven, some being multiple.

III. The number of axes is infinite. It will be convenient to speak of this as the reducible type.

3. If the number of axes is infinite, there exists either a plane or a quadric cone (according as the common factor of (3) is linear or quadratic), such that any direction of ρ in the plane or the cone gives an axis of the vector.

Conversely, that a quadratic vector have an infinite number of axes, it is sufficient that it have six distinct axes on a quadric cone, proper or degenerate. For suppose the number of axes finite. Let the coördinate system be so taken that no axis lies in the plane $z = 0$. If the first two equations (3) have six solutions on a quadric, the remaining three are linearly related.[2] That is, the seventh axis of the vector lies in the same plane with the two lines of intersection of $z = 0$ and $Z = 0$, contrary to hypothesis. Hence the number of axes cannot be finite.

4. The axes are not altered by adding to the vector a term of the form $t\rho$, where t is a scalar. This is geometrically obvious, for if ρ and $F(\rho)$ are parallel, extending $F(\rho)$ in the direction ρ will not disturb the parallelism. Analytically, t must, in the present case, be a linear form in x, y, z. If we write

$$t = px + qy + rz \tag{4}$$

p, q, and r being constants, the addition of $t\rho$ to $F(\rho)$ is the same as putting $X + x(px + qy + rz)$ for X, with similar expressions put for Y and Z. If we make these substitutions in (3), the equations are invariant.[3]

It is equally obvious that the axes of a vector are not altered if we multiply it by any non-vanishing scalar constant.

---

2 Salmon, Higher plane curves, Art. 24, 1st Ed.

3 This is directly seen if we introduce vector multiplication, (see part II, below), for equations (3) are equivalent to the vector equation $V\rho F\rho = 0$. We may change $F\rho$ into $F\rho + t\rho$ at will, since ρ^2 is a scalar.

The results of Arts. 2–4 are evidently applicable with slight modification to homogeneous vectors of any degree, the components being polynomials.

VECTORS OF THE FIRST OR GENERAL TYPE.

5. Theorem I. A quadratic vector of type I is completely determined by its axes, aside from a constant non-vanishing multiplier h and an additive term $t\rho$.

The truth of the theorem would, on the proviso that the axes are independent, appear from a count of the scalar constants involved. For the three quadratic forms X, Y, Z involve eighteen scalars. The multiplier h, and the three scalars which determine t, leave fourteen. The seven axes, if independent, involve fourteen.

That the axes may, in fact, be assigned arbitrarily, I shall show by expressing $F(\rho)$ in terms of its axes by means of a vector equation.

The theorem further implies that, no matter what special relations exist among the axes of $F(\rho)$, provided they are distinct and of finite number, any other vector having the same axes may be written $hF(\rho) + t\rho$.

To prove the theorem, let any seven axes be chosen, distinct, with no six on any quadric cone. Let vectors in these seven directions be denoted by $\beta_1, \beta_2, \ldots \beta_7$. We may, without loss of generality, suppose that β_1, β_2, and β_3 are not coplanar, and, at the same time, that β_4, β_5, and β_6 are not coplanar. For the seven axes cannot all lie in the same plane, because such a plane, with any other plane, would constitute a degenerate quadric cone, contrary to hypothesis. Accordingly, let any three non-coplanar axes be called β_1, β_2, and β_3. The remaining four axes cannot all lie in the same plane, for, with the plane of β_1 and β_2, it would constitute a quadric cone. Let any three which are not coplanar be taken from the four and called β_4, β_5, and β_6.

I shall now determine the constants p, q, and r, of the linear form t, so that β_1, β_2, and β_3 shall be zeros of the vector $F(\rho) + t\rho$; that is, so that we have, simultaneously,

$$F(\beta_1) + t\beta_1 = 0, \quad F(\beta_2) + t\beta_2 = 0, \quad F(\beta_3) + t\beta_3 = 0. \tag{5}$$

Let the components of β_1, β_2, β_3 be given by

$$\beta_1 = ib_{11} + jb_{12} + kb_{13}, \quad \beta_2 = ib_{21} + jb_{22} + kb_{23}, \quad \beta_3 = ib_{31} + jb_{32} + kb_{33}.$$

Let the determinant of the components be denoted by (123), i. e.

$$(123) = \begin{vmatrix} b_{11}, & b_{12}, & b_{13} \\ b_{21}, & b_{22}, & b_{23} \\ b_{31}, & b_{32}, & b_{33} \end{vmatrix} \tag{6}$$

This determinant is not zero, i. e., by hypothesis β_1, β_2, β_3 are not coplanar. That these β's are axes of $F(\rho)$ is equivalent to writing

$$F(\beta_1) = c_1\beta_1;\ F(\beta_2) = c_2\beta_2;\ F(\beta_3) = c_3\beta_3 \tag{7}$$

c_1, c_2, c_3 being constants; whence (4) and (5) yield

$$\begin{aligned} c_1 + pb_{11} + qb_{12} + rb_{13} &= 0 \\ c_2 + pb_{21} + qb_{22} + rb_{23} &= 0 \\ c_3 + pb_{31} + qb_{32} + rb_{33} &= 0 \end{aligned} \tag{8}$$

three linear equations in the three unknowns p, q, r. Since (123) does not vanish, the solution is uniquely possible. Let the values of p, q, r thus determined be p_0, q_0, r_0, and write

$$F_0(\rho) = F(\rho) + t_0\rho$$

The relations (5) are then equivalent to

$$F_0(\beta_1) = 0, \quad F_0(\beta_2) = 0, \quad F_0(\beta_3) = 0.$$

The vector $F_0(\rho)$ is most simply expressed in a new coördinate system given by writing $\rho = \beta_1 x_1 + \beta_2, x_2 + \beta_3, x_3$, equivalent to

$$x_1 = \frac{(23\rho)}{(123)}, \quad x_2 = \frac{(31\rho)}{(123)}, \quad x_3 = \frac{(12\rho)}{(123)}, \tag{9}$$

where (23ρ) denotes the determinant [4]

$$\begin{vmatrix} b_{21}, & b_{22}, & b_{23} \\ b_{31}, & b_{32}, & b_{33} \\ x\ , & y\ , & z \end{vmatrix}$$

with similar meaning for (31ρ) and $12\rho)$. If we write β_1 for ρ we have simultaneously $x_2 = 0$ and $x_3 = 0$. Since $F_0(\beta_1)$ vanishes in all its components, no component can, in the new coördinate system, contain any term in x_1^2. Similarly, no component can contain any term

[4] Using scalar products of three vectors, we may write $x_1 = \dfrac{S\beta_2\beta_3\rho}{S\beta_1\beta_2\beta_3}$ and similarly for x_2 and x_3.

in x_2^2 nor in x_3^2. If the original components of $F_0(\rho)$ be called X_0, Y_0, and Z_0, that is

$$F_0(\rho) = iX_0 + jY_0 + kZ_0,$$

the change of coördinates may be analytically represented by

$$\begin{aligned} X_0 &= X + x(p_0x + q_0y + r_0z) = a_{11}x_2x_3 + a_{12}x_3x_1 + a_{13}x_1x_2 \\ Y_0 &= Y + y(p_0x + q_0y + r_0z) = a_{21}x_2x_3 + a_{22}x_3x_1 + a_{23}x_1x_2 \\ Z_0 &= Z + z(p_0x + q_0y + r_0z) = a_{31}x_2x_3 + a_{32}x_3x_1 + a_{33}x_1x_2, \end{aligned} \tag{10}$$

where the nine a's are constants to be determined. If three vectors α_1, α_2, and α_3 be defined by

$$\alpha_1 = ia_{11} + ja_{21} + ka_{31}, \quad \alpha_2 = ia_{12} + ja_{22} + ka_{32}, \quad \alpha_3 = ia_{13} + ja_{23} + ka_{33}, \tag{11}$$

it is obvious that the transformation (10) is equivalent to

$$F_0(\rho) = \alpha_1x_2x_3 + \alpha_2x_3x_1 + \alpha_3x_1x_2 \tag{12}$$

Taking this result as one step in the demonstration of theorem I, we note that the form of the right member is determined when the choice of axes β_1, β_2, and β_3 has been made. In other words, any two vectors, alike in having β_1, β_2, β_3 for axes can, by a proper choice of p, q, and r, be thrown into the form (12), and will then differ in the vectors α but not otherwise.

Consider next the disposition to be made of the α's that β_4, β_5, and β_6 may be axes of $F_0(\rho)$ and therefore of $F(\rho)$. If $F_0(\beta_4)$ is a scalar multiple of β_4, the determinant of the coefficients of the three vectors β_4, β_5 and $F_0(\beta_4)$ vanishes. Abbreviate this determinant by $(45F_{04})$. Similarly $(45F_{05})$ vanishes if $F_0(\beta_5)$ is parallel to β_5. Advancing cyclically the subscripts 4, 5, 6, we have in this manner six necessary conditions

$$\begin{aligned} (45F_{04}) &= 0, \quad (56F_{05}) = 0, \quad (64F_{06}) = 0, \\ (45F_{05}) &= 0, \quad (56F_{06}) = 0, \quad (64F_{04}) = 0, \end{aligned} \tag{13}$$

They are also sufficient; for, pairing the six relations in a different manner,

$$\begin{aligned} (45F_{04}) &= 0, \quad (56F_{05}) = 0, \quad (64F_{06}) = 0, \\ (64F_{04}) &= 0, \quad (45F_{05}) = 0, \quad (56F_{06}) = 0, \end{aligned}$$

we see that the two equations of the first column require that $F_0(\beta_4)$ shall, at the same time, lie in the plane of β_4, β_5, and the plane of β_6,

β_4. That is, $F_0(\beta_4)$ coincides with β_4 aside from a scalar factor. The sufficiency of the conditions we may, if we prefer, show analytically, by using components b_{41}, b_{42}, b_{43}, for β_4, etc.; and remembering that (456) does not vanish.

From (12) and (9) we have

$$(123)^2 F_0(\rho) = a_1(31\rho)\,(12\rho) + a_2(12\rho)\,(23\rho) + a_3(23\rho)\,(31\rho), \qquad (14)$$

By writing respectively β_4 and β_5 for ρ in this last equation, we obtain the two equations of the first column of (13) in the form [5]

$$\begin{aligned}(45a_1)\,(314)\,(124) + (45a_2)\,(124)\,(234) + (45a_3)\,(234)\,(314) &= 0\\ (45a_1)\,(315)\,(125) + (45a_2)\,(125)\,(235) + (45a_3)\,(235)\,(315) &= 0\end{aligned} \qquad (15)$$

two linear homogeneous equations in the three quantities $(45a_1)$, $(45a_2)$, and $(45a_3)$. The two-row determinants from the matrix of the coefficients cannot all be zero. For, identically,

$$\begin{aligned}\begin{vmatrix}(124)\,(234), & (234)\,(314)\\ (125)\,(235), & (235)\,(315)\end{vmatrix} &\equiv (234)\,(235)\,[(124)\,(315) - (125)\,(314)]\\ &\equiv (234)\,(235)\,(415)\,(123)\end{aligned} \qquad (16)$$

whence, with similar transformations [6] for the other two-row determinants, and remembering that (123) does not vanish, we see that the simultaneous vanishing of the three determinants would be equivalent to

$$(234)\,(235)\,(415) = 0,\ (314)\,(315)\,(425) = 0,\ (124)\,(125)\,(435) = 0 \qquad (17)$$

We cannot have (234) and (314) both zero, for β_3 and β_4 were taken distinct in direction. We may, however, satisfy the first two conditions by assuming $(234) = (315) = 0$. But (124) cannot vanish with (234), (125) cannot vanish with (315), and (435) cannot vanish with (315), since no four axes can lie in one plane. Similarly we may exclude all other combinations of vanishing factors, one from each of the three equations.

---

[5] Using products of vectors, we multiply both sides of (14) by the quaternion $\beta_4\beta_5$ and take scalars.

[6] Using scalar products, the bracketed terms $= S\beta_1\beta_2(\beta_4 S\beta_3\beta_1\beta_5 - \beta_5 S\beta_3\beta_1\beta_4) = S\beta_1\beta_2 V(V\beta_3\beta_1 V\beta_5\beta_4) = S\beta_1\beta_2\beta_3 S\beta_1\beta_5\beta_4 = (123)\,(154)$.

Equations (15) therefore determine the three quantities $(45a_1)$, $(45a_2)$, and $(45a_3)$, in terms of a constant of proportionality, which I shall denote by k_6, as follows,

$$(45a_1)=k_6\begin{vmatrix}(124)\ (234), & (234)\ (314)\\ (125)\ (235), & (235)\ (315)\end{vmatrix}=k_6(123)\ (234)\ (235)\ (415), \quad (18)$$

$$(45a_2)=k_6\begin{vmatrix}(234)\ (314), & (314)\ (124)\\ (235)\ (315), & (315)\ (125)\end{vmatrix}=k_6(123)\ (314)\ (315)\ (425), \quad (19)$$

$$(45a_3)=k_6\begin{vmatrix}(314)\ (124), & (124)\ (234)\\ (315)\ (125), & (125)\ (235)\end{vmatrix}=k_6(123)\ (124)\ (125)\ (435), \quad (20)$$

These equations are changed into one another by cyclic advancement of the numbers 1, 2, and 3. By advancing cyclically the numbers 4, 5, 6 we may obtain two other sets, of three equations each, sufficient to determine the quantities

$$(56a_1),\quad (56a_2),\quad (56a_3);\ \text{and}\ (64a_1),\quad (64a_2),\quad (64a_3);$$

respectively in terms of two other constants of proportionality k_4 and k_5.

These nine relations enable us to write $(F_0\rho)$ in terms of $\beta_1, \beta_2 \ldots \beta_6$ and the constants k_4, k_5, k_6, by means of a vector equation. Consider the determinant of the coefficients of the three vectors β_4, β_5, and $F_0(\rho)$, which we may abbreviate $(45F_0\rho)$. By (14) we have

$$(123)^2\ (45F_0\rho)=(45a_1)\ (31\rho)\ (12\rho)+(45a_2)\ (12\rho)\ (23\rho) +(45a_2)\ (23\rho),\ (31\rho), \quad (21)$$

whence, comparing with (15), we have

$$(123)^2\ (45F_0\rho)=k_6\begin{vmatrix}(31\rho)\ (12\rho), & (12\rho)\ (23\rho), & (23\rho)\ (31\rho)\\ (314)\ (124), & (124)\ (234), & (234)\ (314)\\ (315)\ (125), & (125)\ (235), & (235)\ (315)\end{vmatrix}. \quad (22)$$

If we agree to write

$$P(\rho)=i\ (31\rho)\ (12\rho)+j\ (12\rho)\ (23\rho)+k\ (23\rho)\ (31\rho), \quad (23)$$

we may conveniently denote the determinant on the right of (22) as $(P_4P_5P\rho)$, since it is the determinant of the coefficients of the three vectors $P(\beta_4)$, $P(\beta_5)$ and $P(\rho)$. By advancing the numbers 4, 5, and 6 we obtain two similar relations and have

$$\begin{aligned}(123)^2\,(45F_0\rho) &= k_6\,(P_4P_5P\rho),\\ (123)^2\,(56F_0\rho) &= k_4\,(P_5P_6P\rho),\\ (123)^2\,(64F_0\rho) &= k_5\,(P_6P_4P\rho),\end{aligned} \tag{24}$$

To collect results, note the identity (which we may obtain by writing out determinants),

$$X_0(456) \equiv b_{41}(56F_0\rho) + b_{51}(64F_0\rho) + b_{61}(45F_0\rho),$$

where b_{41}, b_{51} and b_{61} are the first or x-components of β_4, β_5, β_6. If this identity, with two similar identities [7] for Y_0 and Z_0, be multipled respectively by i, j, and k, and the results added, the vectorial identity is obtained

$$(456)\,F_0(\rho) = \beta_4(56F_0\rho) + \beta_5(64F_0\rho) + \beta_6(45F_0\rho), \tag{25}$$

Values determined by equations (24), (equivalent to the six equations (13), necessary and sufficient that β_4, β_5, and β_6 shall be axes), introduced in (25), give

$$(123)^2\,(456)\,F_0(\rho) = k_4\beta_4(P_5P_6P\rho) + k_5\beta_5(P_6P_4P\rho) + k_6\beta_6(P_4P_5P\rho), \tag{26}$$

The form of this result shows that, on the one hand $F_0(\beta_1)$, $F_0(\beta_2)$, and $F_0(\beta_3)$ vanish, (because $P(\beta_1)$, $P(\beta_2)$, and $P(\beta_3)$ vanish), while on the other hand we have

$$F_0(\beta_4) = \beta_4 \cdot \frac{k_4(P_4P_5P_6)}{(123)^2\,(456)}, \tag{27}$$

with similar expressions for $F_0(\beta_5)$ and $F_0(\beta_6)$. As a step in the demonstration of theorem I, we note that two vectors, alike in having $\beta_1, \beta_2, \ldots \beta_6$ for axes, can be thrown into the form (26), and will then differ in the constants k_4, k_5, k_6, but not otherwise.

It remains to dispose of k_4, k_5 and k_6 so that β_7 shall be an axis. Let β_7 be expressed in terms of β_4, β_5, and β_6 by an identity like (25), viz.

$$(456)\,\beta_7 = \beta_4(567) + \beta_5(647) + \beta_6(457), \tag{28}$$

If β_7 is an axis, $F_0(\beta_7) = h\beta_7$ where h is some constant; whence, writing β_7 for ρ in (26),

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7 The three are equivalent to the well-known vector identity, (β, λ, μ, ν, being any four vectors), $\beta S\lambda\mu\nu \equiv \lambda S\mu\nu\beta + \mu S\nu\lambda\beta + \nu S\lambda\mu\beta$.

$$(123)^2(456)k\beta_7 = k_4\beta_4(P_5P_6P_7) + k_5\beta_5(P_6P_4P_7) + k_6\beta_6(P_4P_5P_7), \quad (29)$$

Comparing corresponding components of (β_7) in (28) and (29) we have, as necessary and sufficient conditions that β_7 shall be an axis,

$$k_4 = \frac{h(567)\ (123)^2}{(P_5P_6P_7)}, k_5 = \frac{h(647)\ (123)^2}{(P_6P_4P_7)}, k_6 = \frac{h(457)\ (123)^2}{(P_4P_5P_7)}, \quad (30)$$

Allowing for the moment that none of the denominators vanish, we may introduce these results in (26) and have, finally,

$$F_0(\rho) = \beta_4 \cdot \frac{h(567)\ (P_5P_6P\rho)}{(456)\ (P_5P_6P_7)} + \beta_5 \cdot \frac{h(647)\ (P_6P_4P\rho)}{(456)\ (P_6P_4P_7)} + \beta_6 \frac{h(457)\ (P_4P_5P\rho)}{(456)\ (P_4P_5P_7)}, \quad (31)$$

If β_7 be written for ρ, the right side reduces to $h\beta_7$ by the identical relation (28). h cannot be zero for $F_0(\rho)$ would vanish and $F(\rho)$ would reduce to the term $t\rho$, contrary to the hypothesis that $F(\rho)$ is of type I. h is otherwise arbitrary and two vectors alike in possessing the axes $\beta_1, \beta_2, \ldots \beta_7$ can differ in the constants h, c_1, c_2, and c_3, that is, in regard to h and the form t, but not otherwise.

I shall now show that none of the denominators in (30) can vanish if the choice of β_7 is consistent with the hypothesis that no six axes lie on a quadric cone; whence the seven axes of (31) are assignable in any manner consistent with that hypothesis. Consider the determinant on the right of (22), or $(P_4P_5P\rho)$. Expanding by the elements of the first row, and developing the minors as in (18), (19), and (20), we have

$$\begin{aligned}(P_4P_5P\rho) = \{&(31\rho)\ (12\rho)\ (234)\ (235)\ (415) \\ &+ (12\rho)\ (23\rho)\ (314)\ (315)\ (425) \\ &+ (23\rho)\ (31\rho)\ (124)\ (125)\ (435)\ \}\ (123)\end{aligned}$$

In the first term on the right, in place of the product of the two factors $(31\rho)\ (234)$, write, identically, $(314)\ (23\rho) + (123)\ (34\rho)$. Then employ successively the two identities

$$(235)\ (415) + (315)\ (425) = (345)\ (125) \text{ and } (31\rho)\ (124) - (12\rho)\ (314) = (123)\ (41\rho)$$

and we have

$$(P_4P_5P\rho) = (123)^2 \{ (12\rho)\,(34\rho)\,(235)\,(415) - (125)\,(345)\,(23\rho)\,(41\rho) \} \quad (32)$$

The expression in braces may be regarded as a function of the four vectors β_1, β_2, β_3, β_4, taken in cyclic order, and of the two vectors ρ and β_5. It may be abbreviated $C_{1234}(5, \rho)$. It is homogeneous and quadratic in each of the six vectors, and vanishes if any two coincide. Therefore, if equated to zero, it denotes a quadric cone through the five vectors β. If we write β_7 instead of ρ, the result cannot vanish, since, by hypothesis, no six axes lie on a quadric cone. That is, the third denominator in (30) does not vanish. Similarly, neither $(P_5P_6P_7)$ nor $(P_6P_4P_7)$ can vanish. Theorem I is therefore proved.

6. We may regard (31) as a normal or model form for a vector of type I. A variety of results follow immediately, either by inspection, or by using identities like (32).

For example, if a quadratic vector $F(\rho)$ has three axes which are coplanar, it may, by the addition of a properly chosen term $\rho(px + qy + rz)$ be reduced to a binomial. For let the coplanar axes be numbered 4, 5, 7, and proceed as in (31). The last term vanishes, and $F_0\rho$ is a sum of scalar multiples of two axes. The resulting binomial vector has β_1, β_2, β_3, and β_6 for zeros.

Conversely, if $F(\rho)$, being of type I, has four zeros, it has the other three axes coplanar. For choose three of the zeros to be the β_1, β_2, β_3 of the foregoing discussion. Determine $F_0(\rho)$ as in (31). β_7 cannot be a zero, for, as has been shown, h cannot be zero. If either β_4, β_5, or β_6 is a zero, we have k_4, k_5, or k_6, respectively, zero, entailing the vanishing of either (457), (567), or (647). That is, a set of three axes lie in a plane, not including one of the four zeros.

Again, let there be three coplanar axes, and number them 1, 2, and 7. By expanding as in (32) we see that each of the denominators in the model form (31) becomes a product of three-row determinants.

If a vector of type I has two sets of coplanar axes, the two sets must have one, and only one, axis in common, since no six lie on any quadric cone. Suppose $(127) = (457) = 0$. (31) becomes, aside from a scalar factor,

$$\beta_4 \cdot \frac{(567) \cdot C_{1235}(6, \rho)}{(126)\,(356)\,(517)} + \beta_5 \cdot \frac{(647) \cdot C_{1236}(4, \rho)}{(124)\,(364)\,(617)} \quad (33)$$

where $C_{1235}(6, \rho)$ vanishes on the cone through β_1, β_2, β_3, β_5 and β_6, and is obtained as in (32). There is nothing to prevent us from inter-

changing the two sets of axes numbered 1, 2, 3 and 4, 5, 6, whence (33) becomes

$$\beta_1 \cdot \frac{(237) \cdot C_{4562}(3, \rho)}{(345)(236)(247)} + \beta_2 \cdot \frac{(317) \cdot C_{4562}(1, \rho)}{(145)(631)(347)} \tag{34}$$

and the two vectors (33) and (34) can, by theorem I, differ only in a scalar factor and a term $(px + qy + rz)\rho$. As an example of the striking relations that hold between the constants p, q, r and the axes, let p, q, and r be determined so that $t\rho$ added to (33) gives a vector equal or parallel to (34). If we write

$$\delta = ip + jq + kr \tag{35}$$

the vector δ thus determined is at right angles to both the axes β_3 and β_6 which do *not* enter into either of the coplanar sets, a consequence of the fact that β_3 and β_6 are zeros of both vectors (33) and (34). More generally, if $F_1(\rho) = hF_2(\rho) + t\rho$, and if F_1 and F_2 have a common zero β, t must vanish if ρ has the direction β, i. e. δ is at right angles to β.[8]

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8 Darboux has pointed out, (loc. cit.) the importance of linear relations of the type (127) = 0 in the solution of differential equations. For example, that the solution of the equation

$$(yZ - zY)dx + (zX - xZ)dy + (xY - yX)dz = 0$$

may be made to depend on that of a Riccati equation, it is necessary and sufficient that we have, in the language of this present paper, three sets of coplanar axes, with one common axis, e. g. (127) = (457) = (367) = 0. If there are four sets of coplanar axes the equation can be integrated by quadratures. On the other hand, if we have three coplanar sets, but not with one common axis, e. g. (127) = (457) = (134) = 0, no general solution of the equation has been obtained.

Another application of the ideas developed in the text is to point transformations. If we regard x_1, x_2, x_3, of (9) as plane homogeneous coördinates, (31) gives the most general quadratic transformation having three singular points β_1, β_2, β_3, and four fixed points β_4, β_5, β_6, β_7.

PART II. REDUCIBLE VECTORS.

7. To obtain typical forms for vectors of the third class, which it will be convenient to consider next, much is gained in simplicity by introducing vector multiplication. If we adopt the Hamiltonian laws for i, j, and k,—

$$ij = k,\ jk = i,\ ki = j;\ ji = -k,\ kj = -i,\ ik = -j,$$
$$i^2 = j^2 = k^2 = -1.$$

it is well known that vector multiplication is distributive with respect to addition and is associative. It is obviously not commutative. The product of two vectors is, in general, partly a scalar, and partly a vector. These two parts of the product are denoted respectively, by the selective symbols S and V. We may verify by direct multiplication that equations (3) are equivalent to the vector equation

$$V\rho F(\rho) = 0 \tag{36}$$

It was shown in Art. 2 that, if the number of axes of $F(\rho)$ is infinite, the left members of equations (3) have a common factor, that is, $V\rho F(\rho)$ consists of a scalar factor multiplied into a vector of lower degree.

I shall now show that reducible quadratic vectors may be thrown into one of three typical forms, according as they possess

(a) A proper cone of axes, every element of the cone being an axis of the vector.

(b) A single plane of axes, every direction in the plane being an axis of the vector.

(c) Two planes of axes, which, as a special case, may be in coincidence.

In each of these cases, there will, in general, be one or more discrete axes not in the plane or cone of axes.

8. The above subdivision of reducible vectors follows readily from certain properties of homogeneous vectors.

Theorem II. If a vector $F_n(\rho)$, whose components are homogeneous polynomials in x, y, and z of degree n, satisfies the identity

$$S\rho F_n(\rho) \equiv 0 \tag{37}$$

it can be written as $V\rho F_{n-1}(\rho)$, where $F_{n-1}(\rho)$ is a vector of degree $n-1$.

For let the components of $F_n(\rho)$ be X, Y, and Z. The identity (37) is equivalent to

$$xX + yY + zZ \equiv 0. \tag{38}$$

When y and z vanish together, x does not in general vanish, hence X must vanish. Therefore X, as a polynomial in x, y, and z, can contain no term in x^n. We may therefore write $X = yw + zv$ where v and w are scalar polynomials of degree $n - 1$. Similarly,

$$Y = zu + xw' \quad \text{and} \quad Z = xv' + yu'.$$

(38) becomes

$$yz(u + u') + zx(v + v') + xy(w + w') \equiv 0. \tag{39}$$

When $x = 0$ neither y nor z are generally zero, hence $u + u'$ vanishes all over the plane $x = 0$. With similar reasoning for y and z we may write

$$u + u' = px, \quad v + v' = qy, \quad w + w' = rz, \tag{40}$$

where, in the case $n = 1$, the factors p, q, and r are necessarily zero, since u, u', etc. are constants, but for larger values of n we may have p, q, and r polynomials of degree $n - 2$. From (39) we now obtain

$$p + q + r \equiv 0 \tag{41}$$

By eliminating u', v', w', and p, we have

$$X = yw + zv, \quad Y = z(u + rx) - xw, \quad Z = -vx - y(u + rx) \tag{42}$$

If, therefore, we write

$$F_{n-1}(\rho) = iP + jQ + kR = i(u + rx) + j(-v) + kw, \tag{43}$$

we find by actual multiplication

$$\begin{aligned} V\rho F_{n-1}(\rho) &= i(yw + zv) + j(zu + zrx - xw) - k(vx + yu + yrx) \\ &= F_n(\rho), \text{ by (42).} \end{aligned}$$

The vector $F_{n-1}(\rho)$ is not uniquely determined, since we may add to it an arbitrary vector term of the form ρt, where t is a scalar polynomial of degree $n-2$.

Theorem III. If a vector $F_n(\rho)$, whose components are homogeneous polynomials in x, y, and z of degree n, satisfies the identity

$$V\rho F_n(\rho) \equiv 0 \tag{44}$$

it can be written in the form ρt, where t is a scalar polynomial of degree $n-1$.

Proof. Identity (44) implies that all directions of ρ are axes of the vector, or that equations (3) become identities for the vector in question. It follows that X vanishes all over the plane $x = 0$, and we may write $X = tx$ where t is a polynomial of degree $n-1$. Similarly, $Y = ty$ and $Z = tz$, the factor t being the same in all three cases, by (3). This proves the theorem.[9]

9. Returning now to the case of a reducible quadratic vector $F(\rho)$, if the common factor of the left members of (3) is a quadratic polynomial which is irreducible, we have

$$VF(\rho) = q\phi\rho \tag{45}$$

where q is the quadratic scalar and $\phi\rho$ is a vector of the first degree in ρ. If we multiply both sides of (45) by ρ and take scalars we have

$$S\rho\phi\rho = 0, \tag{46}$$

because $S\cdot\rho V\rho F(\rho) = S\cdot\rho^2 F(\rho) = 0$. Therefore by theorem II

$$\phi\rho = V\alpha\rho \tag{47}$$

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[9] Theorems similar to II and III may be proved by Euler's theorem for any vectors whose components are homogeneous of the same degree. In general, if $F(\rho)$ is of degree n, we have the identity

$$F(\rho) \equiv V\rho F_{n-1}(\rho) + \nabla s,$$

where s is a scalar function of degree $n+1$ and ∇ is the differential operator

$$i\frac{\delta}{\delta x} + j\frac{\delta}{\partial y} + k\frac{\delta}{\partial z}$$

From this, theorem II follows at once, s being zero. The vector $F_{n-1}(\rho)$ may always be taken parallel to $V\nabla F(\rho)$. See Phil. Mag., **29** (May 1915), p. 704.

where α is a constant vector. We may thus write (45) in the form

$$V\rho \{ F(\rho) - q\alpha \} = 0 \tag{48}$$

By theorem III the vector in braces is a scalar multiple of ρ, and it is of the second degree, giving

$$F(\rho) = q\alpha + \rho t \tag{49}$$

where t is a linear form in x, y, and z. It is evident that α is an axis of the right member of (49). The cone $q = 0$ is a cone of axes. (49) may be regarded as a normal form for type (a) of reducible quadratic vectors. In vectorial language, a scalar quadratic form may always be written $S\rho\theta\rho$, where $\theta\rho$ is a linear vector, and a linear form t may always be written $S\delta\rho$ where δ is a constant vector. (49) then becomes

$$F(\rho) = \alpha S\rho\theta\rho + \rho S\delta\rho. \tag{50}$$

No change occurs in the order of reasoning in case the quadratic form s, that is $S\rho\theta\rho$, is reducible to a product of linear factors. By Art. 3, if a quadratic vector possesses two sets of three coplanar axes, the six axes being distinct, it is a reducible vector. Any vector in either of the two planes containing the sets of three must be an axis of the vector, which may be written, as a normal form (c),

$$F(\rho) = \alpha S\beta_1\rho\beta_2\rho + \rho S\delta\rho \tag{51}$$

where β_1 and β_2 are constant vectors normal to these two planes, giving $S\beta_1\rho$ and $S\beta_2\rho$ two linear forms.

If the common factor of the left members of (3) is a linear polynomial, we shall have, instead of (45),

$$V\rho F(\rho) = S\beta\rho \cdot G(\rho) \tag{52}$$

where $S\beta\rho$ is the linear factor and $G(\rho)$ is, consequently, a quadratic vector. Multiplying both sides by ρ and equating the scalar parts,

$$S\rho G(\rho) = 0, \tag{53}$$

whence by theorem II

$$G(\rho) = V\rho\phi\rho, \tag{54}$$

where $\phi\rho$ is a linear vector. We may thus write (52) in the form

$$V\rho[F(\rho) - \phi\rho \cdot S\beta\rho] = 0 \tag{55}$$

By theorem III the vector in brackets is a scalar multiple of ρ, and it is of the second degree, giving

$$F(\rho) = \phi\rho \cdot S\beta\rho + \rho S\delta\rho, \tag{56}$$

where $S\delta\rho$ is, as before, a linear form. It is evident that the three axes of the linear vector $\phi\rho$ are axes of the right member and that any vector in the plane $S\beta\rho = 0$ is an axis. It is also clear that this type (b) of reducible quadratic vector contains, as does (a), the more special type (c) as a limiting case, since $\phi\rho$ may itself be reducible, i. e., have an infinite number of axes. By Art. 3, a sufficient condition that a quadratic vector shall be of type (b) is that it shall possess four distinct axes in the same plane.

Finally, if the left members of (3) vanish identically, theorem III shows that the quadratic vector $F(\rho)$ is of the form $\rho S\delta\rho$, and it may then be regarded as a limiting case of either (a), (b), or (c). Collecting results, any reducible vector $F(\rho)$ of the second degree may be written in one of the three type forms

$$\begin{array}{ll} (a) & \alpha S\rho\theta\rho + \rho S\delta\rho \\ (b) & \phi\rho S\beta\rho + \rho S\delta\rho \\ (c) & \alpha S\beta_1\rho S\beta_2\rho + \rho S\delta\rho \end{array} \tag{57}$$

10. The following negative theorem is occasionally useful,—

Theorem IV. If, for a given quadratic vector $F(\rho)$, seven distinct axes can be found such that no six lie on a quadric cone, and if $V\rho F(\rho)$ does not vanish identically, $F(\rho)$ is not reducible.

Proof. If the vector is reducible of type (a), it cannot consist merely of its last term $\rho S\delta\rho$, since, by hypothesis, $V\rho F(\rho)$ does not vanish identically. Its only axes are the vector α, with the cone of axes $S\rho\theta\rho = 0$. Whence it is not possible to choose seven not having six on this cone. Similar reasoning applies to (c). If the vector is reducible of type (b), we may suppose ϕ to possess not more than three distinct axes, for if so it could be written as (c).[10] Whence it is im-

10 For $V\rho F(\rho) = S\beta\rho \cdot V\rho\phi\rho$. If $\phi\rho$ has more than three distinct axes, $V\rho\phi\rho$ has a linear factor, by reasoning parallel to that of Art. 3.

possible to choose seven distinct axis not having four in the plane $S\beta\rho = 0$. This plane, with the plane of two other axes, constitutes a quadric cone.

There are no other possibilities; that is, it is never possible to choose, for a reducible quadratic vector, seven distinct axes without six on a quadratic cone, (excluding the quadratic vector $\rho S\delta\rho$). This is the theorem.

When the axes are not already known, we may test $F(\rho)$ for reducibility by resolving $V\rho F(\rho)$ into scalar components in any convenient manner, and examining these scalars for common factors according to any of the well-known geometric or algebraic processes for detecting reducible polynomials.

When, by any method, a scalar factor has been found for $V\rho F(\rho)$, we throw $F(\rho)$ into the proper type form, by the processes of Art. 9.

It is of value, in theoretical investigations, to have tests for reducibility not requiring resolution into components. These are always possible. For example, if $F(\rho)$ is of type (a) or type (c), $V\rho F(\rho)$ is always in the plane at right angles to α. Hence if ρ_1, ρ_2, and ρ_3 are any three values of ρ we must have

$$S(V\rho_1 F\rho_1)\,(V\rho_2 F\rho_2)\,(V\rho_3 F\rho_3) = 0 \qquad (58)$$

The further study of these tests leads naturally to the use of differential operators, and lies beyond the purpose of this paper.

PART III. VECTORS WITH MULTIPLE AXES.

11. I shall now suppose that a quadratic vector is given possessing three known axes, distinct and diplanar. With the notation and ideas of the first part of this paper, we may suppose the vector to be thrown into the form (12), by the addition of a term $\rho S\delta\rho$, that is $t\rho$.

Let the three vectors α_1, α_2, α_3 be expressed in terms of the three axes β_1, β_2, β_3 by identities like (25), e. g.

$$\alpha_1 S\beta_1\beta_2\beta_3 = \beta_1 S\beta_2\beta_3\alpha_1 + \beta_2 S\beta_3\beta_1\alpha_1 + \beta_3 S\beta_1\beta_2\alpha_1, \tag{59}$$

the scalar of the product of three vectors being, sign excepted, the determinant of their components. If we adopt the notation

$$A_{11} = \frac{S\beta_2\beta_3\alpha_1}{S\beta_1\beta_2\beta_3}, \quad A_{21} = \frac{S\beta_3\beta_1\alpha_1}{S\beta_1\beta_2\beta_3}, \quad A_{12} = \frac{S\beta_2\beta_3\alpha_2}{S\beta_1\beta_2\beta_3}, \text{ etc.,} \tag{60}$$

we shall have $F_0(\rho)$ in the form

$$\begin{aligned} F_0(\rho) = {} & \beta_1(A_{11}x_2x_3 + A_{12}x_3x_1 + A_{13}x_1x_2) \\ & + \beta_2(A_{21}x_2x_3 + A_{22}x_3x_1 + A_{23}x_1x_2) \\ & + \beta_3(A_{31}x_2x_3 + A_{32}x_3x_1 + A_{33}x_1x_2), \end{aligned} \tag{61}$$

where the nine A's are constants to be determined.

If β_1 is a double axis, the cubic cones (3) have the same tangent plane at the element β_1, or else have a double line at β_1. By taking polars,[11] the vector $V\rho F(\rho)$ gives

$$V\rho F_0(\beta_1) + V\beta_1\beta_2(A_{22}x_3 + A_{23}x_2) + V\beta_1\beta_3(A_{32}x_3 + A_{33}x_2), \tag{62}$$

because x_2 and x_3 vanish when β_1 is put for ρ. But $F_0(\beta_1)$ vanishes. Hence, that we should have, at most, one polar plane for the three cubic cones (3), it is necessary and sufficient that the determinant

$$\begin{vmatrix} A_{22}, & A_{23} \\ A_{32}, & A_{33} \end{vmatrix} \tag{63}$$

11 That is, differentiating $V\rho F\rho$ and putting β_1 for ρ after the differentiation; we have, as the polar vector, $V\rho'F\rho + V\rho\{\beta_1(A_{11}x_2'x_3 + A_{11}x_2x_3') + \ldots \text{ etc.}\}$, which, writing β_1 for ρ, and dropping accents, gives (62).

shall vanish. β_1 is thus a multiple axis of $F(\rho)$ when, and only when, this condition is satisfied.

We may now suppose β_4, β_5, and β_6 to be three more axes, as in Art. 5. If we assume, as before, that no quadric cone can be passed through the six vectors $\beta_1 \ldots \beta_6$, and that β_4, β_5 and β_6 are diplanar, the investigation of Art. 5 is valid through (27). The vanishing of (63) must therefore be equivalent to a relation between the constants of proportionality k_4, k_5, and k_6. To obtain this relation we have first to write α_2 in terms of β_4, β_5, β_6, by an identity like (28),

$$(456)\alpha_2 = \beta_4(56\alpha_2) + \beta_5(64\alpha_2) + \beta_6(45\alpha_2) \tag{64}$$

For A_{22} we then have

$$\begin{aligned}(123)\,(456)A_{22} &= (456)\,(31\alpha_2), \text{ by (60)},\\ &= (314)\,(56\alpha_2) + (315)\,(64\alpha_2) + (316)\,(45\alpha_2), \text{ by (64)},\\ &= (314)k_4(123)\,(315)\,(316)\,(526)\\ &+ (315)k_5(123)\,(316)\,(314)\,(624)\\ &+ (316)k_6(123)\,(314)\,(315)\,(425),\end{aligned} \tag{65}$$

by using the value of $(45\alpha_2)$ from (19), with two similar expressions for $(56\alpha_2)$ and $(64\alpha_2)$. The result may be most simply expressed by taking a vector κ such that its components along $V\beta_5\beta_6$ etc. are k_4, k_5, k_6, that is

$$(456)\kappa = k_4V\beta_5\beta_6 + k_5V\beta_6\beta_4 + k_6V\beta_4\beta_5 \tag{66}$$

We then have, multiplying both sides by β_2 and taking scalars,

$$\begin{aligned}(456)S\kappa\beta_2 &= S(k_4\beta_2\beta_5\beta_6 + k_5\beta_2\beta_6\beta_4 + k_6\beta_2\beta_4\beta_5)\\ &= -\,[k_4(562) + k_5(642) + k_6(452)]\end{aligned}$$

because the scalar of the product of three vectors is the negative of their determinant. (65) may now be written

$$A_{22} = -\,(314)\,(315)\,(316)S\kappa\beta_2 \tag{67}$$

By similar reasoning

$$A_{33} = -\,(124)\,(125)\,(126)S\kappa\beta_3 \tag{68}$$

To obtain A_{23}, it is more practicable to use, in (20), the determinant

form of $(45a_3)$, with two similar determinants for $(56a_3)$ and $(64a_3)$; we have

$$A_{23}(123)\,(456) = (31a_3)\,(456), \text{ by (60)},$$
$$= (314)\,(56a_3)+(315)\,(64a_3)+(316)\,(45a_3), \text{ identically},$$

whence, using (20) as above indicated, the last expression is the same as the determinant

$$\begin{vmatrix} k_4(314), & (314)\,(124), & (124)\,(234) \\ k_5(315), & (315)\,(125), & (125)\,(235) \\ k_6(316), & (316)\,(126), & (126)\,(236) \end{vmatrix} \tag{69}$$

If we expand by the elements of the third column we find

$$A_{23}(123)\,(456) = (124)\,(234)\,(315)\,(316)\,[k_5(126)-k_6(125)]+\ldots+\ldots; \tag{70}$$

but, by (66), $k_5 = -S\kappa\beta_5$ and $k_6 = -S\kappa\beta_6$, giving

$$\begin{aligned} k_5(126)-k_6(125) &= -S\kappa\beta_5(126) + S\kappa\beta_6(125) \\ &= +S\kappa[\beta_5 S\beta_1\beta_2\beta_6 - \beta_6 S\beta_1\beta_2\beta_5] \\ &= S\kappa V(V\beta_1\beta_2 V\beta_6\beta_5), \text{ identically}, \\ &= S\kappa[\beta_1 S\beta_2\beta_6\beta_5 - \beta_2 S\beta_1\beta_6\beta_5] \\ &= S\kappa\beta_1(256) - S\kappa\beta_2(156); \end{aligned} \tag{71}$$

the second and the third terms on the right of (70) may be similarly transformed, being obtained from the first term by advancing the numbers 4, 5, 6. If we collect the coefficients of $S\kappa\beta_1$ we therefore have

$$(124)\,(234)\,(315)\,(316)\,(256) + (125)\,(235)\,(316)\,(314)\,(264) + (126)\,(236)\,(314)\,(315)\,(245).$$

In the first of these three terms, make the identical transformations

$$(234)\,(315) = (123)\,(345) + (235)\,(314)$$

and in the third term,

$$(236)\,(315) = (123)\,(365) + (235)\,(316).$$

If the resulting five terms are grouped into those with the factor (123) and those without it we have

$$(123)\,[(124)\,(345)\,(316)\,(256) + (126)\,(365)\,(314)\,(245)] + (314)\,(316)\,(235)\,\{(124)\,(256) + (125)\,(264) + (126)\,(245)\}.$$

The expression in braces vanishes, for it is identically equal to

$$(122)\,(456)$$

by a transformation like (64). The coefficient of (123) may be written

$$(453)\,(613)\,(562)\,(142) - (452)\,(612)\,(563)\,(143) \qquad (72)$$

by a mere rearrangement. But this is the same as $C_{4561}\,(23)$ by the notation of Art. 6.

Collecting the coefficients of $-S\kappa\beta_2$ we have precisely similar transformations to make, except that, in the last factor of every term, β_1 is written for β_2, i. e. 1 for 2. Thus the coefficient of (123) in the result is

$$(124)\,(345)\,(316)\,(156) + (126)\,(365)\,(314)\,(145),$$

while the other terms contain the factor $(121)\,(456)$ and vanish. By a slight rearrangement, the two above terms may be written

$$(453)\,(613)\,(561)\,(142) - (451)\,(612)\,(563)\,(143) \qquad (73)$$

which may be regarded as derivable from (72), symbolically, by the operation $1\cdot\frac{\delta}{\delta 2}$; this is the same as saying, geometrically, that a quadric cone through the vectors β_4, β_5, β_6, β_1, and β_3 is denoted by $C_{4561}(\rho, 3) = 0$, and that the polar of β_1 with respect to C be taken at β_2. If the tangent plane to this cone at β_1, obtained by polarization of C, be denoted by $T_{4561}(\rho_1, 3) = 0$, we shall naturally write (73) as $T_{4561}(_{12}, 3)$. The relation between T and C is most easily expressed by the operator ∇, thus

$$T_{4561}(_{12}, 3) = S\beta_1\nabla'\cdot C_{4561}(2', 3), \qquad (74)$$

where, as indicated by the accents, ∇ operates on β_2 alone, or, if we prefer, ρ is written for β_2 before the operation. These results enable us to write, from (70), (cancel (123)),

$$A_{23}\,(456) = S\kappa\beta_1\cdot C_{4561}(2, 3) - S\kappa\beta_2\cdot T_{4561}(_{12}, 3) \qquad (75)$$

In a similar manner we may obtain any other A with double subscript in terms of κ. Thus

$$A_{32}\,(456) = S\kappa\beta_1 \cdot C_{4561}(3, 2) - S\kappa\beta_3 \cdot T_{4561}(_{13}, 2), \tag{76}$$

where, as before, T may be obtained from C by writing ρ for β_3 and operating by $-S\beta_1\nabla$, and afterwards writing β_3 for ρ.

Before putting for the A's their values in the determinant (63), it will be well for the sake of symmetry of form, to transform A_{22} as follows,

$$\begin{aligned}(456)\,A_{22} &= -\,(456)\,(314)\,(315)\,(316)S\kappa\beta_2, \quad \text{by} \quad (67),\\ &= S\kappa\beta_2 \cdot (314)\,(316)[(451)\,(563) - (453)\,(561)], \text{ identically},\\ &= S\kappa\beta_2 \cdot [(453)\,(613)\,(561)\,(143) - (451)\,(613)\,(563)\,(143)],\\ &= S\kappa\beta_2 \cdot T_{4561}(_{31}, 3), \quad \text{by} \quad (72),\end{aligned} \tag{77}$$

where $T_{4561}(_{31}, 3)$ denotes the result of polarizing $C_{4561}(\rho, 3)$ with respect to β_1 and β_3. It is evident that any 'T' which is a function of five vectors only can be similarly transformed. Thus

$$(456)\,A_{33} = S\kappa\beta_3 \cdot T_{4562}(_{12}, 1) \tag{78}$$

The determinant (63) may now be written

$$\begin{vmatrix} & S\kappa\beta_2 \cdot T_{4561}(_{31}, 3);\ S\kappa\beta_1 \cdot C_{4561}(2, 3) \\ & \qquad\qquad - S\kappa\beta_2 \cdot T_{4561}(_{12}, 3) \\ S\kappa\beta_1 \cdot C_{4561}(3, 2) - S\kappa\beta_3 \cdot T_{4561}(_{13}, 2);\ & S\kappa\beta_3 \cdot T_{4562}(_{12}, 1) \end{vmatrix} \tag{79}$$

whose vanishing determines that β_1 shall be a double axis, and clearly requires that κ shall lie on a quadric cone. The constant C, and its derived constant 'T,' are found at once when the six axes are assigned.

12. A second method for obtaining a general condition for a multiple axis is to start with (26), which, by (32), becomes

$$(456)F_0(\rho) = k_4\beta_4 C_{1235}(6, \rho) + k_5\beta_5 C_{1236}(4, \rho) + k_6\beta_6 C_{1234}(5, \rho). \tag{80}$$

We may make β_4 a double axis by so choosing k_4, k_5, and k_6 that the vector $V\rho F(\rho)$, polarized [12] with respect to β_4 and equated to zero,

12 That is, forming the polar vector as in the note to Art. 11, we write β_4 for ρ after the operation. The easiest way to form the polar vector in this case is to multiply (80) by ρ and operate by $S\rho'\nabla$.

yields not more than one distinct scalar equation; for this is the same as saying that the cubic cones (3) are either tangent at β_4 or have a double line there. The resulting vector equation is

$$V\beta_4\beta_5 \cdot k_5 T_{1236}(4, {}_{4\rho}) + V\beta_4\beta_6 \cdot k_6 T_{1234}(5, {}_{4\rho}) + V\rho\beta_4 \cdot k_4 C_{1235}(6, 4) = 0. \tag{81}$$

If this is equivalent to one scalar equation, and if we put for ρ, in succession, any two distinct vectors, the coefficients of $V\beta_4\beta_5$ and of $V\beta_4\beta_6$ in the two results must be in proportion. Take as the two vectors β_5 and β_6. The proportionality of the coefficients is given by the vanishing of the determinant whose elements are these coefficients, viz.

$$\begin{vmatrix} k_5 T_{1236}(4, {}_{45}) - k_4 C_{1235}(6, 4), & k_6 T_{1234}(5, {}_{45}) \\ k_5 T_{1236}(4, {}_{46}) \quad , & k_6 T_{1234}(5, {}_{46}) - k_4 C_{1235}(6, 4) \end{vmatrix} \tag{82}$$

whose vanishing determines that β_4 shall be a double axis. As a verification, we may note that if we write $k_4 = -S\kappa\beta_4$, etc., this result differs from (79) only in the numbering of the axes.

13. The two methods above given for obtaining a multiple axis depended upon applying the theory of polars to the function C of six vectors. The determinants (79) and (82) are, in fact, symmetrical functions of the five single axes, and give a general relation between κ, the double axis, and the other five. They may be transformed in many ways by identities similar to those already used. It is desirable, however, to have a normal form for a quadratic vector with a multiple axis in which the tangent plane to the cones (3) at the double axis shall appear explicitly. This may easily be found as follows,— Start with the general normal form (31). Write $\beta_4 = m\alpha + n\beta_5$. By this substitution $(P_4P_5P\rho)$ becomes, with the aid of (32),

$$m_2(P\alpha P_5 P\rho) + mn(123)_2 \{ (12\rho)\,(35\rho)\,(235)\,(\alpha 15) - (125)\,(3\alpha 5)\,(23\rho)\,(51\rho)\}$$

The expression in braces is quadratic in β_1, β_2, β_3 and ρ. It expresses, by its vanishing, a quadric cone through β_1, β_2, β_3, and β_5, with the tangent plane at β_5 given by $(5\alpha\rho) = 0$; to prove this, take the polar with respect to β_5,

$$(125)\,(35\rho)\,(235)\,(\alpha 15) - (125)\,(3\alpha 5)\,(235)\,(51\rho)$$

but this is equal to (125) (235) (351) ($5a\rho$) by the identity

$$(35\rho)\ (a15) - (3a5)\ (51\rho) = (351)\ (5a\rho).$$

If we therefore agree to write, in keeping with the notation already used,

$$C_{1235}({}_5a, \rho) = (12\rho)\ (35\rho)\ (235)\ (a15) - (125)\ (3a5)\ (23\rho)\ (51\rho), \quad (83)$$

the writing of $ma + n\beta_5$ for β_4 gives

$$(P_4P_5P\rho) = m^2(PaP_5P\rho) + mn\ (123)^2C_{1235}({}_5a, \rho)$$

and, similarly,

$$(P_4P_5P_7) = m^2(PaP_5P_7) + mn\ (123)^2C_{1235}({}_5a, 7).$$

The factors (457) and (456) become m ($a57$) and m ($a56$). Hence the coefficient of β_6 in (31) becomes

$$\frac{(a57)\ [m(PaP_5P\rho) + n(123)^2C_{1235}({}_5a, \rho)]}{(a56)\ [m(PaP_5P_7) + n(123)^2C_{1235}({}_5a, \rho)]}$$

which, if m approaches zero, approaches the limit

$$\frac{(a57)C_{1235}({}_5a, \rho)}{(a56)C_{1235}({}_5a, 7)} \quad (84)$$

Expressions like the right of (83), while of geometrical significance, are sometimes less convenient than determinants, (or scalar products), like (PaP_5P_7). We might have kept the latter form of work by writing at the start, (by (23)),

$$P(ma + n\beta_5) = m^2Pa + n^2P_5 + mnPa_5, \quad (85)$$

where Pa_5 has been written for

$$Pa_5 = i\ [(31a)\ (125) + (315)\ (12a)] + j\ [(12a)\ (235) + (125)\ (23a)] + k\ [(23a)\ (315) + (235)\ (31a)] \quad (86)$$

If we attach a similar meaning to any other P with double subscript (i. e., the result of polarizing $P(\rho)$ with respect to two vectors), the coefficient of β_6 in (31) approaches, by the same reasoning as before, the limit

$$\frac{(a57)\ (Pa_5P_5P\rho)}{(a56)\ (Pa_5P_5P_7)} \tag{87}$$

The factor $(Pa_5P_5P\rho)$ differs from $C_{1235}({}_5a,\ \rho)$ only in the presence of a factor $(123)^2$. (Cf. (32)).

Considering the remaining terms of (31), the coefficient of a after the substitution of $ma + n\beta_5$ for β_4 is

$$\frac{(567)\ (P_5P_6P\rho)}{(a56)\ (P_5P_6P_7)}$$

By the aid of (85), the coefficient of β_5 may be written

$$\frac{n(567)\ (P_5P_6P\rho)}{m(a56)\ (P_5P_6P_7)}$$
$$+\frac{[m(6a7) - n(567)]}{m(a56)}\frac{[m^2(PaP_6P\rho) + mn(Pa_5P_6P\rho) + n^2(P_5P_6P\rho)]}{[m^2(PaP_6P_7) + mn(Pa_5P_6P_7) + n^2(P_5P_6P_7)]}$$

which, if we let m approach zero, approaches the limit

$$\frac{(567)\ [(P_5P_6P\rho)\ (Pa_5P_6P_7) - (P_5P_6P_7)\ (Pa_5P_6P\rho)] + (6a7)\ (P_5P_6P_7)\ (P_5P_6P\rho)}{(a56)\ (P_5P_6P_7)^2}$$

a result rendered more compact by the identity

$$(P_5P_6P\rho)\ (Pa_5P_6P_7) - (P_5P_6P_7)\ (Pa_5P_6P\rho) = (P_5P_6Pa_5)\ (P_6P_7P\rho)$$

Collecting results, we find that as m approaches zero, (31) approaches the limiting form

$$\frac{k_7a(567)\ (P_5P_6P\rho)}{(a56)\ (P_5P_6P_7)}$$
$$+\frac{k_7\beta_5[(567)\ (P_5P_6Pa_5)\ (P_6P_7P\rho) + (6a7)\ (P_5P_6P_7)\ (P_5P_6P\rho)]}{(a56)\ (P_5P_6P_7)^2}$$
$$+\frac{k_7\beta_6(a57)\ (Pa_5P_5P\rho)}{(a56)\ (Pa_5P_5P_7)}, \tag{88}$$

a normal form for a quadratic vector having the vectors β_1, β_2, β_3, β_6, and β_7 as ordinary axes, but β_5 a double axis, the cones (3) being tangent to the plane $(a5\rho) = 0$ along the vector β_5. We may, if we wish, verify directly by polarization that $V\rho F\rho = 0$ gives at most one

scalar equation when operated on by $S\rho'\nabla$, if $\rho = \beta_5$. The cones (3) do not, in general, have a double line at β_5. When, however, we choose one axis of coördinates along β_5, at least one of these cones passes twice through the double axis.

14. A fourth method for multiple axes, while in some respects less convenient, in that the tangent plane to (3) at the double axis is less explicitly contained in the result, brings the present discussion into close relation to the theory of point transformations; — instead of (23) take

$$Q(\rho) = i(12\rho)^2 + j(12\rho)\,(2\alpha'\rho) + k(2\alpha'\rho)\,(\alpha'1\rho) \qquad (89)$$

This vector evidently has β_1 for an axis. By polarization, it is obvious that it has β_2 for a double axis, with the plane $(2\alpha'\rho) = 0$ tangent to the cones (3) or meeting them twice at β_2. If, therefore, we replace P by Q in the normal form (31) we shall have β_1, β_4, β_5, β_6, and β_7 as single axes, β_2 as a double axis, and $(2\alpha'\rho) = 0$ the tangent plane to (3) at β_2. Furthermore, the result will be the most general quadratic vector satisfying these conditions, aside from an additive term $\rho S\delta\rho$, for it can otherwise differ from (88) only in the numbering of the axes. Any vector with a multiple axis differs from another with the same multiple axis only in the direction of the tangent plane to (3) at that axis, a multiplicative constant, and the term $\rho S\delta\rho$; provided the five single axes also coincide.

15. These methods for multiple axes may be employed simultaneously to obtain two, or three, distinct double axes. Thus to form a vector having β_1, β_6, and β_7 for single axes, β_2 and β_5 for double axes, we have only to write Q instead of P in (88). The vector α may, in (89), be the same vector as in (88), i. e., it may be the line of intersection of the tangent planes to the cones (3) at β_2 and β_5, — with the obvious restriction that this line does not itself coincide with an axis. In general we may, if we wish, take α and α' any two vectors such that, with the five axes, no six vectors lie on a quadric cone.

More symmetrically, let the vector have β_1, β_2, β_3, for single axes, and two other double axes. By writing $\beta_7 = m\alpha + n\beta_6$ in (88) and letting m approach zero we easily find

$$\alpha(P_5P_6P\alpha_5)\,(P_5P_6P\alpha_6)\,(P_5P_6P\rho) + \beta_5(P_5P_6P\alpha_5)^2\,(P_6P\alpha_6P\rho) + \beta_6(P_5P_6P\alpha_6)^2\,(P\alpha_5P_5P\rho), \qquad (90)$$

as a normal form for a quadratic vector with two double axes, β_5 and β_6, which may be renumbered at our convenience. The vector α is here the line of intersection of the tangent planes to (3) at the double axes.

The symmetrical form (90) is possible only when the three similar axes are not in the same plane. If they are coplanar, some of the methods previously described may be used instead, e. g., the function Q may be used.[13]

For three double axes we might write Q for P in this last result, it being now necessary to take, in general, α distinct from the α' of (89). For a symmetrical formula, however, we shall best return to (61), and impose upon the nine A's conditions that β_1, β_2, β_3 shall all be double axes, viz. that the determinant (63) shall vanish together with two others obtained by advancing subscripts. If we wish the single axis to appear explicitly, we shall most easily begin with the general normal form (31), writing $m_1\beta_4 + n_1\beta_1$ instead of β_4, $m_2\beta_5 + n_2\beta_2$ instead of β_5, and $m_3\beta_6 + n_3\beta_3$ instead of β_6. As m_1, m_2, and m_3 approach zero we have the limit

$$\frac{\beta_1(237)\,(P_{25}P_{36}P\rho)}{(123)\,(P_{25}P_{36}P_7)} + \frac{\beta_2(317)\,(P_{36}P_{14}P\rho)}{(123)\,(P_{36}P_{14}P_7)} + \frac{\beta_3(127)\,(P_{14}P_{25}P\rho)}{(123)\,(P_{14}P_{25}P_7)}, \quad (91)$$

as a normal form for a quadratic vector having β_1, β_2, β_3, for double axes, β_7 for a single axis, and the planes $(14\rho) = 0$, $(25\rho) = 0$, and

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13 As a simple example leading to a vector of the type (91), let it be required to investigate whether the equations

$$\frac{dx}{xy} = \frac{dy}{yz} = \frac{dz}{zx}$$

can be integrated by quadratures. The integration depends upon that of the partial differential equation $SF\rho\nabla u = 0$ where

$$F\rho = ixy + jyz + kzx$$

and hence upon

$$S\rho d\rho F\rho = 0.$$

We easily find that i, j, and k are double axes of $F\rho$ and that $i + j + k$ is the single axis. The tangent planes to (3) at i, j, and k are found by taking polars of $V\rho F\rho$ thus,—

$$S\rho'\nabla \cdot V\rho F\rho = V\rho' F\rho + V\rho[i(x'y + xy') + j(y'z + yz') + k(z'x + zx')] = 0,$$

which, putting $\rho = i$, $y = z = 0$, $x = 1$, gives the single scalar equation $z' = 0$. Similarly we have $x' = 0$ and $y' = 0$ at j and k, respectively. Thus the vector $F\rho$ is a limiting form of a type having three sets of coplanar axes, in the three coördinate planes, the sets not possessing an axis common to all. Hence the equation is not reducible to quadratures by any known process, (Cf. note to Art. 6), nor even to a Riccati equation.

$(36\rho) = 0$, the tangent planes to the cones (3) at the double axes. Any other vector having the same axes and the same tangent planes can differ from the above at most by a multiplicative constant and an additive term $\rho S\delta\rho$.

With regard to the case of three double axes (91) gives the most general form of such a quadratic vector. For, by its method of derivation, it is always possible whenever the three double axes are not coplanar. But no irreducible vector can have three distinct multiple axes in the same plane: a fact we would perhaps guess from the stand-point of Art. 3, if the same ideas apply to vectors with multiple axes; for the plane of the three double axes, taken twice, would constitute a quadric cone. It is more conclusive to prove directly. The following method of attack is, moreover, applicable to a variety of cases.

Let two axes of a quadratic vector be β_1 and β_2. Let β_3 be some third vector, not necessarily an axis, but such that (123) does not vanish. Let a suitable term $\rho S\delta\rho$ be added to the vector so as to make zeros of β_1 and β_2, (by two equations like (8)). With the notation (9), the resulting vector can have no terms in x_1^2 or in x_2^2, but may have terms in x_3^2. If we now expand as in (61), writing B_1, B_2, and B_3, for the coefficients of x_3^2, we shall have

$$\begin{aligned} &\beta_1(A_{11}x_2x_3 + A_{12}x_3x_1 + A_{13}x_1x_2 + B_1x_3^2) \\ +\ &\beta_2(A_{21}x_2x_3 + A_{22}x_3x_1 + A_{23}x_1x_2 + B_2x_3^2) \\ +\ &\beta_3(A_{31}x_2x_3 + A_{32}x_3x_1 + A_{33}x_1x_2 + B_3x_3^2). \end{aligned} \tag{92}$$

This is evidently a form to which any quadratic vector with two known axes can be reduced with ease.

The necessary and sufficient conditions that this vector possess a third axis coplanar with β_1 and β_2, but distinct, are

$$A_{33} = 0, \quad A_{13} \text{ not zero}, \quad A_{23} \text{ not zero}. \tag{93}$$

For directions in the plane of β_1 and β_2, but not along β_1 nor β_2, are given by

$$x_3 = 0, \quad x_1 \text{ not zero}, \quad x_2 \text{ not zero} \tag{94}$$

If $x_3 = 0$, and (92) lies in the plane of β_1 and β_2, we must thus have necessarily $A_{33} = 0$. The direction of (92) is then

$$\beta_1 A_{13} + \beta_2 A_{23}. \tag{95}$$

which is therefore an axis; coinciding with β_1 or β_2 if either A_{13} or A_{23} vanish. Therefore conditions (93) are both necessary and sufficient.

To introduce double axes we now form the polar vector of $V\rho\, F\rho$ by operating with $S\rho'\nabla$ or its equivalent, yielding

$$\begin{aligned}
&V\rho' \{ \beta_1(A_{11}x_2x_3 + A_{12}x_3x_1 + A_{13}x_1x_2 + B_1x_3^2) \\
&\quad + \beta_2(A_{21}x_2x_3 + A_{22}x_3x_1 + A_{23}x_1x_2 + B_2x_3^2) \\
&\quad + \beta_3(A_{31}x_2x_3 + A_{32}x_3x_1 + \quad + B_3x_3^2) \} \\
&+ V\rho\{\beta_1[A_{11}(x'_2x_3 + x_2x'_3) + A_{12}(x'_3x_1 + x_3x'_1) + A_{13}(x'_1x_2 + x_1x'_2) \\
&\qquad + 2B_1x'_3x_3] \\
&\quad + \beta_2[A_{21}(x'_2x_3 + x_2x'_3) + A_{22}(x'_3x_1 + x_3x'_1) + A_{23}(x'_1x_2 + x_1x'_2) \\
&\qquad + 2B_2x'_3x_3] \\
&\quad + \beta_3[A_{31}(x'_2x_3 + x_2x'_3) + A_{32}(x'_3x_1 + x_3x'_1) + \\
&\qquad + 2B_3x'_3x_3]. \qquad (96)
\end{aligned}$$

That a direction ρ shall be a double axis is the same as saying that the polar vector equated to zero shall yield at most one scalar equation. At β_1 we have $x_2 = x_3 = 0$, leading to the vanishing of the determinant (63), independently of the B's. But this leads to $A_{32} = 0$, because A_{23} is not zero, while A_{33} vanishes. In a similar manner we have $A_{31} = 0$. That is, all three A's of the third line of (92) are zero.

To make the direction $\beta_1A_{13} + \beta_2A_{23}$ a double axis, we write, in the polar vector, $\rho = \beta_1A_{13} + \beta_2A_{23}$, $x_1 = A_{13}$, $x_2 = A_{23}$, $x_3 = 0$. We have also $\rho' = \beta_1x'_1 + \beta_2x'_2 + \beta_3x'_3$. Substituting these values and dropping accents we must have

$$\begin{aligned}
&V\{\beta_1A_{13} + \beta_2A_{23}\} \{\beta_1[A_{11}A_{23}x_3 + A_{12}A_{13}x_3 + A_{13}(A_{23}x_1 + A_{13}x_2)] \\
&\qquad + \beta_2[A_{21}A_{23}x_3 + A_{22}A_{13}x_3 + A_{23}(A_{23}x_1 + A_{13}x_2)] \\
&+ V\{\beta_1x_1 + \beta_2x_2 + \beta_3x_3\} \{\beta_1A_{13}^2A_{23} + \beta_2A_{13}A_{23}^2\} = 0 \qquad (97)
\end{aligned}$$

equivalent to a single scalar equation in ρ. The coefficient of $V\beta_3\beta_1$ is $x_3A_{13}^2A_{23}$, so that if neither A_{13} nor A_{23} vanish the required scalar equation must be $x_3 = 0$. This is the same as saying that the tangent plane to the cones (3) at the required axis must be the plane of β_1 and β_2, (in so far as these cones do not possess double lines there). The coefficient of $V\beta_2\beta_3$ yields nothing new. In the coefficient of $V\beta_1\beta_2$, on the other hand, we have terms in x_3 together with

$$A_{13}A_{23}(A_{23}x_1 - A_{13}x_2). \qquad (98)$$

That the coefficient of $V\beta_1\beta_2$ shall be a multiple of x_3 is thus impossible. This completes the proof.

16. From the foregoing results on double axes it appears that any quadratic vector having less than seven distinct axes but no triple axis may be written in one of the three following normal forms,—

(a) If there is but one double axis, write Q for P in (31). The tangent plane to (3) at the double axis can pass, at most, through one of the single axes. Any other single axis may be taken as a zero of $Q(\rho)$. The remaining four single axes cannot all lie in the same plane. Any three which are diplanar may be numbered 4, 5, and 6, to correspond with (31).

(b) If there are just two double axes, write Q for P in (88). Choose either double axis to be a zero of Q. The tangent plane to (3) at this double axis can pass at most through one of the single axes. Choose either of the other single axes to be the second zero of $Q(\rho)$. At the double axis not already taken, the tangent plane to (3) can pass, at most, through one of the two remaining single axes. The one through which it passes, (if either), must be taken as β_7. Otherwise, the choice of numbering is arbitrary.

(c) If there are three double axes, (91) is always possible.

17. It remains to consider vectors with less than six distinct axes, one of which is of multiplicity three or greater. For triple axes we have at our disposal a variety of methods, analogous to those used above for double axes. We may, for example, assign a relation to connect the constants A of (31) in order that β_1 may be a triple axis.

It may well happen that all four elements of the determinant (63) are zero. If so, the cones (3) all have a double line at β_1, which is consequently a quadruple axis. I shall assume, for the present, that such is not the case.

This possibility excluded, the cubic cone

$$S\lambda\rho F\rho = 0, \tag{99}$$

where λ is a constant vector, (so that $S\lambda\rho F\rho$ is linearly related to the left members of (3)), will not have a double line at β_1 for all values of λ. To say that β_1 is a triple axis of $F\rho$ is therefore equivalent to saying that all cones obtained by varying λ, exclusive of those with double lines at β_1, will osculate along β_1. Or again, all these cones give the same curvature for normal sections at any point on an element β_1. The most direct way to express this condition is to say that $d\nu$ is independent of λ, where ν is a vector of unit length normal to the cones at a point on an element β_1, and $d\rho$ is any vector in the

tangent plane. If $d\rho$ is parallel to ρ, $d\nu$ vanishes, because we are dealing with cones; it is sufficient, therefore, to consider any other one direction of $d\rho$ in the tangent plane, preferably the direction $V\rho\nu$, the direction of maximum curvature. If we write $d\nu = \chi d\rho$, $V\rho\nu$ is an axis of the linear vector function χ, and $\chi V\rho\nu = gV\rho\nu$, where g is the maximum curvature of a normal section of the cone.[14] But this curvature is equal to the "divergence" of the unit vector ν at the point on the cone. For, by definition of divergence,

$$\operatorname{div} \nu = -S\nabla\nu = -S\nu\chi\nu - Su\chi u - S\epsilon\chi\epsilon, \tag{100}$$

where u and ϵ are unit vectors along ρ and $V\rho\nu$, respectively. But $S\nu\chi\nu$ vanishes, because the differential of a unit vector is always perpendicular to the unit vector itself. And χu vanishes because we deal with cones. As above, $\chi\epsilon = g\epsilon$, giving, (because $\epsilon^2 = -1$),

$$g = -S\nabla\nu \tag{101}$$

The necessary and sufficient conditions for a triple axis may accordingly be stated: if β is a triple axis of $F\rho$, a unit vector normal to the cone (99) and its divergence have at most one determinate direction and one determinate numerical value, respectively, independent of λ.

18. The direction of the normal is found by operating with ∇ on $S\lambda\rho F\rho$. If we put, for convenience

$$\sigma = \nabla S\lambda\rho F\rho \tag{102}$$

so that

$$\nu = \frac{\sigma}{T\sigma} \tag{103}$$

we have, from (101),

$$\begin{aligned} g &= -S\nabla\left(\frac{\sigma}{T\sigma}\right) \\ &= -\frac{S\nabla\sigma}{T\sigma} + \frac{S\sigma\nabla T\sigma}{T^2\sigma} \\ &= \frac{\sigma^2 S\nabla\sigma + S\sigma\varphi\sigma}{T^3\sigma}, \end{aligned} \tag{104}$$

14 For a more detailed examination of χ, see Phil. Mag., June, 1902, p. 576, and Feb., 1903, p. 187.

because $\nabla T\sigma = \phi'\nu$, when $d\sigma = \phi d\rho$.[15] The numerator of (104) may be written in a number of remarkable forms. It is, for example, the Hessian of the ternary form $S\lambda\rho F\rho$, multiplied by a factor independent of $F\rho$. This follows from the fact that H is Hamilton's m-invariant for the function ϕ; that is, if i, j, k, are ANY three diplanar vectors,

$$H = \frac{S\phi i\phi j\phi k}{Sijk} \tag{105}$$

Choose, as three convenient vectors, ρ, σ, and $V\rho\sigma$. Then

$$H = \frac{S\phi\rho\phi\sigma\phi V\rho\sigma}{S\rho\sigma V\rho\sigma} \tag{106}$$

By a well-known expansion [16] we have

$$\phi V\rho\sigma = -S\nabla\sigma\cdot V\rho\sigma - V\phi\rho\sigma - V\rho\phi\sigma \tag{107}$$

If we write this value for $\phi V\rho\sigma$, and multiply out, remembering that $\phi\rho$ is parallel to σ because we deal with homogeneous functions, while $S\rho\sigma$ vanishes on the cone, we have H equal to the numerator of (104) aside from a factor which is a constant multiple of ρ^2.

More important for our present purpose than this connection with the Hessian, is the fact that the numerator of (104) can be obtained by differential operations performed directly upon the vector $V\rho F\rho$. For, taking the first term of this numerator,

$$S\nabla\sigma = \nabla^2 S\lambda\rho F\rho = S\lambda\nabla^2\rho F\rho, \tag{108}$$

because ∇^2 is a scalar and commutative. As to the second term of the same numerator,

$$S\sigma\phi\sigma = -\sigma^2 S\nu\phi\nu, \text{ identically.} \tag{109}$$

If we let γ be the direction of ν at the point on the cone, we have

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15 Proof: $dT^2\sigma = -d\sigma^2 = -2S\sigma d\sigma = -2S\sigma\phi d\rho$. Hence $\nabla T^2\sigma = 2\phi'\sigma = 2T\sigma\nabla T\sigma$, and, dividing by $2T\sigma$, we have $\nabla T\sigma = \phi' U\sigma = \phi'\nu$.

16 Hamilton, Elements of Quaternions, Art. 350. Hamilton's m'' is the same as $-S\nabla\sigma$, and ϕ is self-conjugate.

$$-S\gamma\phi\gamma = -S\nu\phi\nu = \frac{d^2}{dh_\gamma} S\lambda\rho F\rho,$$

the second derivative of the ternary form $S\lambda\rho F\rho$ along the normal, more conveniently written $S^2\gamma\nabla \cdot S\lambda\rho F\rho$. By the commutative property of $S^2\gamma\nabla$ we thus have

$$S\sigma\phi\sigma = \sigma^2 S\lambda S^2\gamma\nabla \cdot \rho F\rho, \tag{111}$$

These results substituted in (104) give, as the greatest curvature of a normal section of the cone at a point where the normal is in the direction γ,

$$g = S\lambda \frac{\gamma^2\nabla^2 - S^2\gamma\nabla}{T\sigma} \rho F\rho \tag{112}$$

where the factor γ^2 is introduced for homogeneity, in order that γ need not be a unit vector. This new numerator thus defines a differential operation upon $V\rho F\rho$.

We may now introduce the conditions for a triple axis. First, σ is, in direction, independent of λ, hence is of the form $\gamma S\eta\lambda$, when β, the axis, is written for ρ. Therefore

$$T\sigma = T\gamma S\eta\lambda, \tag{114}$$

and, in order that λ may cancel from the expression for g, it is necessary and sufficient that

$$(\gamma^2\nabla^2 - S^2\gamma\nabla) V\rho F\rho$$

shall be parallel to η when β is put for ρ after the differentiation.

But this condition may be still further simplified. Let α be the direction which $V\rho\gamma$ takes when β is written for ρ, so that α, β, and γ form a rectangular system. Therefore, if they are taken of unit length,

$$\nabla^2 = -S^2\alpha\nabla - S^2\beta\nabla - S^2\gamma\nabla$$

while $S^2\beta\nabla$ vanishes if β be put for ρ after the differentiation, because β is an axis. This gives

$$\gamma^2\nabla^2 - S^2\gamma\nabla = +S^2\alpha\nabla,$$

when applied to $V\rho\boldsymbol{F}\rho$ at a point on the element β. Now if $d\boldsymbol{F}\rho = \Phi(\rho, d\rho)$,

$$\begin{aligned} S^2\alpha\nabla\cdot V\rho\boldsymbol{F}\rho &= S\alpha\nabla\cdot[V\alpha\boldsymbol{F}\rho + V\rho\Phi(\alpha, \rho)] \\ &= 2V\alpha\Phi(\alpha, \rho) + 2V\rho\boldsymbol{F}\alpha, \end{aligned}$$

and writing β for ρ, we have twice the vector

$$V\alpha\Phi(\alpha, \beta) + V\beta\boldsymbol{F}\alpha \tag{114}$$

which is the polar vector of $V\rho\boldsymbol{F}\rho$ with regard to α at β.

Returning now to the direction η, this is the direction which the polar vector of $V\rho\boldsymbol{F}\rho$ takes at β, and is perpendicular to β. Also, $\Phi(\alpha, \beta)$ is parallel to β. If we agree to write

$$\eta = V\beta\pi, \quad S^2\alpha\nabla\cdot V\rho\boldsymbol{F}\rho = V\beta\tau, \tag{115},$$

β being put for ρ after the differentiation, the parallelism of these two vectors is expressed by

$$VV\beta\pi V\beta\tau = 0$$

which, by a simple expansion, reduces to

$$S\beta\pi\tau = 0, \tag{116}$$

which is both necessary and sufficient that g shall be independent of λ. We may sum up the foregoing investigation of triple axes in the rule:— Let the polar vector of $V\rho\boldsymbol{F}\rho$ be $V\rho'\boldsymbol{F}\rho + V\rho\Phi(\rho', \rho)$. If β is a double axis, and β be written for ρ, the polar vector takes the form $S\gamma\rho'V\beta\pi$; while if α be written for ρ and β for ρ', the polar vector takes the form $V\beta\tau$. The necessary and sufficient condition for β to be a triple axis is $S\beta\pi\tau = 0$.

19. It now becomes a simple matter to apply this rule to (61). If β_1 be a double axis, we have, as already shown, $A_{22}A_{33} - A_{23}A_{32} = 0$. If β_1 be written for ρ, the polar vector, by (62), becomes

$$V\beta_1\beta_2(A_{22}x_3 + A_{23}x_2) + V\beta_1\beta_3(A_{32}x_3 + A_{33}x_2) \tag{117}$$

The normal direction is thus the normal to the plane determined by

$$A_{22}x_3 + A_{23}x_2 = A_{32}x_3 + A_{33}x_2 = 0 \tag{118}$$

By (9) these equations are equivalent to

$$A_{22}S\beta_1\beta_2\rho + A_{23}S\beta_3\beta_1\rho = A_{32}S\beta_1\beta_2\rho + A_{33}S\beta_3\beta_1\rho = 0 \qquad (119)$$

Let γ and β_1 be supposed unit vectors; and let c_2 and c_3 be two constants defined by

$$A_{22}V\beta_1\beta_2 + A_{23}V\beta_3\beta_1 = c_2\gamma, \quad A_{32}V\beta_1\beta_2 + A_{33}V\beta_3\beta_1 = c_3\gamma \qquad (120)$$

The equations which determine the tangent plane then become

$$c_2S\gamma\rho = c_3S\gamma\rho = 0 \qquad (121)$$

and the polar vector for β_1 becomes, by (117),

$$V\beta_1(c_2\beta_2 + c_3\beta_3)S\gamma\rho \qquad (122)$$

The required vector π is thus given by

$$\pi = c_2\beta_2 + c_3\beta_3 \qquad (123)$$

The direction of α is $V\beta_1\gamma$, which is certainly determined by

$$c'_2V\beta_1(A_{22}V\beta_1\beta_2 + A_{23}V\beta_3\beta_1) + c'_3V\beta_1(A_{32}V\beta_1\beta_2 + A_{33}V\beta_3\beta_1), \qquad (124)$$

since, under the present hypothesis, the four A's are not all zero; c_2^1 and c_3^1 being any two new constants such that (124) does not vanish.

To find τ we have, by the rule, to write β_1 for ρ, (instead of for ρ'), in the polar vector, which then becomes

$$V\beta_1F\rho + V\rho\beta_1(A_{12}x_3 + A_{13}x_2) + V\rho(c_2\beta_2 + c_3\beta_3)S\gamma\rho \qquad (125)$$

On writing α for ρ, $S\gamma\rho$ vanishes; and the remaining terms are at right angles to β_1. It is therefore obvious that, in (124), we may neglect any component along β_1. Multiplying out, (124) gives

$$(\beta_1^2 - \beta_1S\beta_1)[(c'_2A_{22} + c'_3A_{32})\beta_2 - (c'_2A_{23} + c'_2A_{33})\beta_3] \qquad (126)$$

By dropping the component along β_1, we see that we may use, instead of the true value of α, the simpler vector

$$(c'_2A_{22} + c'_3A_{32})\beta_2 - (c'_2A_{23} + c'_3A_{33})\beta_3 \qquad (127)$$

The required vector τ is the result of writing the above vector for ρ in

$$F\rho - \rho(A_{12}x_3 + A_{13}x_2), \tag{128}$$

namely, the vector into which β_1 is multiplied in (125), $S\gamma\rho$ vanishing. By (9), the substitution of ρ for (127) is equivalent to

$$x_1 = 0, \quad x_2 = c'_2A_{22} + c'_3A_{32}, \quad x_3 = -c'_2A_{23} - c'_3A_{33}. \tag{129}$$

This gives, for $F\alpha$,

$$F\alpha = -(c'_2A_{22} + c'_3A_{32})(c'_2A_{23} + c'_3A_{33})(A_{21}\beta_2 + A_{31}\beta_3),$$

neglecting the β_1 component. Substituting values in (128), and collecting coefficients, the condition $S\beta_1\pi\tau = 0$ may be arranged as

$$\begin{aligned}
&c'_2{}^2\{c_2(A_{23}{}^2A_{12} - A_{23}A_{13}A_{22} + A_{31}A_{22}A_{23}) - c_3(A_{22}{}^2A_{13} - A_{22}A_{12}A_{23} \\
&\qquad + A_{21}A_{22}A_{23})\} \\
&+c'_2c'_3\{c_2(A_{22}A_{33} + A_{23}A_{32})(+A_{31} - A_{13}) + 2A_{23}A_{12}A_{23} \\
&\qquad - c_3(A_{22}A_{33} + A_{23}A_{32})(+A_{21} - A_{12}) + 2A_{22}A_{32}A_{13}\} \\
&+c'_3{}^2\{c_2(A_{33}{}^2A_{12} - A_{33}A_{13}A_{32} + A_{31}A_{32}A_{33}) - c_3(A_{32}{}^2A_{13} - A_{32}A_{12}A_{33} \\
&\qquad + A_{21}A_{32}A_{33})\} = 0.
\end{aligned} \tag{130}$$

The three expressions in braces are easily seen to be equivalent, when none of the four elements of (63) are zero, in virtue of the equations

$$\frac{c_2}{c_3} = \frac{A_{22}}{A_{32}} = \frac{A_{23}}{A_{33}} \tag{131}$$

In any case, all three expressions in braces must vanish, since the choice of constants c'_2 and c'_3 is arbitrary.

Considering various cases that may arise, the vanishing of any one of the four elements of (63) entails the vanishing of one of the constants c_2 or c_3, provided the vector $F\rho$ is not reducible. For example, suppose $A_{33} = 0$. The vanishing of (63) entails $A_{23}A_{32} = 0$. If $A_{23} = 0$, the vanishing of the third line of (130) gives $A_{13} = 0$ and $F\rho$ is reducible; whence we have $A_{32} = 0$ and so $c_3 = 0$. Similarly we may show that the vanishing of either A_{32}, A_{22}, or A_{23}, entails the vanishing of one of the constants c_2 or c_3. Accordingly we have only three possibilities,—

(a). Neither c_2 nor c_3 is zero. The three expressions in braces are equivalent.

(b) $c_2 = 0$, that is, $A_{22} = A_{23} = 0$. The first and second expressions in braces vanish of themselves. The third gives

$$A_{32}(A_{32}A_{13} - A_{12}A_{33} + A_{21}A_{33}) = 0. \tag{132}$$

(c) $c_3 = 0$, that is $A_{32} = A_{33} = 0$. The second and third expressions in braces vanish of themselves. The first gives

$$A_{23}(A_{23}A_{12} - A_{13}A_{22} + A_{31}A_{22}) = 0. \tag{133}$$

There are no other possible cases for irreducible quadratic vectors, these conditions are, then, necessary and sufficient for a triple axis β_1. In subcases (b) and (c), the meaning of the condition is, geometrically, as follows: let $V\rho F\rho = 0$ define three cubic cones by separation into components along $V\beta_1\beta_2$, $V\beta_2\beta_3$, and $V\beta_3\beta_1$. The second of these always has a double line at β_1. If $c_2 = 0$, the first also has β_1 for a double line. The condition (132) then requires than the tangent plane to the third cone shall touch one sheet of the first. It is easy to show that it also touches one sheet of the second, whence a triple axis. We have a similar meaning for (133).

20. We may note in passing that the rule developed in Art. 18 for detecting double and triple axes is applicable to vectors of any degree,— with, however, one important modification. In the quadratic case, $S^2\alpha\nabla V\rho F\rho$ can be obtained from the polar vector. When $F\rho$ is of higher degree the rule may read:

General rule for Double and Triple Axes of Vectors.

If β is a double axis of a vector $F\rho$, (homogeneous in ρ), the derived vector $S\rho'\nabla \cdot V\rho F\rho$ takes the form $S\gamma\rho' V\beta\pi$ when β is written for ρ after the differentiation; and the second derived vector $S^2\beta\gamma\nabla \cdot V\rho F\rho$ takes the form $V\beta\tau$ when β is written for ρ after the differentiation. The axis β is of multiplicity higher than two if (and only if) $S\beta\pi\tau = 0$.

These two formal conditions may be combined in one vector equation as follows. Let $d\rho$ and $\delta\rho$ be two independent differentials of ρ. The vector

$$V(\delta V\rho F\rho)\,(d^2V\rho F\rho) \tag{134}$$

must vanish if, after the differentiation, β is written for ρ, and $d\rho$ is

any vector perpendicular to $\nabla S\rho F\rho\delta\rho$, (i. e. $d\rho$ is any vector tangent to the cones (3)). The vanishing is independent of the value of $\delta\rho$, corresponding to the arbitrary constants of (130).

The reader familiar with Hamilton's theory of linear vectors will perceive the condition for a double axis to be identical with the condition that the vector function $\nabla S\delta\rho\rho F\rho$, linear in $\delta\rho$, shall have two roots of its symbolic cubic equal to zero.

21. To obtain a normal form for a quadratic vector having β_1 as a triple axis, we may apply the method of limits to (31). Let β_4 be replaced by $m\beta_1 + n\beta_4$ and let n approach zero. In the limit we obtain, somewhat after the manner of (91),

$$\beta_1 \frac{h(567)\,(P_5P_6P\rho)}{(156)\,(P_5P_6P_7)} + \beta_5 \frac{h(617)\,(P_6P_{14}P\rho)}{(156)\,(P_6P_{14}P_7)} + \beta_6 \frac{h(157)\,(P_{14}P_5P\rho)}{(156)\,(P_{14}P_5P_7)} \tag{135}$$

This vector has β_1 for a double axis, with the tangent plane to the cones (3) that of β_1 and β_4. We may now cause β_7 to approach β_1 as a limit by writing, instead of β_7, the vector $\beta_1 + x\beta_4 + cx^2\beta_7$, and at the same time putting hx for h. The constant c is arbitrary except as noted below. When x approaches zero, the vector (135) approaches the limit

$$\beta_1 \frac{h(P_5P_6P\rho)}{(P_5P_6P_{14})} + \beta_5 \frac{h(614)\,(P_6P_{14}P\rho)}{(156)\,(P_6P_{14}[P_4 + cP_{17}])} + \beta_6 \frac{h(154)\,(P_{14}P_5P\rho)}{(156)\,(P_{14}P_5[P_4+cP_{17}]}, \tag{136}$$

which may be taken as the required normal form. With regard to the denominators of the second and third terms, the cone

$$(P\rho P_{14}P_4) + c(P\rho P_{14}P_{17}) = 0 \tag{137}$$

passes through the axes β_1, β_2, and β_3, with tangent plane at β_1 the plane of β_1 and β_4, and with principal curvature determined by the constant c. If the left of (137) be abbreviated $D\rho$, the denominators of the terms in question become, respectively, (156) D_6 and $-(156)D_5$. These denominators do not vanish, therefore, so long as the cone (137) does not contain as elements either β_5 or β_6. But this cone is

an osculating cone to the cones (3) at the element β_1, and will therefore not contain β_5 or β_6 so long as the quadratic vector $F\rho$ is not reducible.

As a verification, the test for a triple axis may be applied to the vector (136) by the rule of Art. 18. The polar vector of $V\rho F\rho$ is

$$V\rho\beta_1 \frac{(P_5P_6P\rho\rho')}{(P_5P_6P_{14})} + V\rho\beta_5 \frac{(614)\ (P_6P_{14}P\rho\rho')}{(156)D_6} + V\rho\beta_6 \frac{(154)\ (P_{14}P_5P\rho\rho')}{-(156)D_5} + V\rho' F\rho,$$

where $F\rho$ has the form (136). Writing β_1 for ρ, the first and fourth terms vanish. To evaluate the other terms note that, by (86),

$$VP_{14}P_1\rho = i(123)^3\,(41\rho), \tag{138}$$

whence, by (23),

$$\begin{aligned}(P_6P_{14}P_1\rho) &= \quad (316)\ (126)\ (123)^3\,(41\rho),\\ (P_{14}P_5P_1\rho) &= -\ (315)\ (125)\ (123)^3\,(41\rho).\end{aligned} \tag{139}$$

The vector π, that is the vector coefficient of β_1, is thus

$$\beta_5 \frac{(614)\ (316)\ (126)}{(156)D_6} - \beta_6 \frac{(154)\ (315)\ (125)}{(156)D_5}, \tag{140}$$

common factors of the two terms being dropped. Writing α for ρ and β_1 for ρ' in the polar vector, the result is

$$V\beta_4\beta_1 + V\beta_1\beta_5 \frac{(614)\ (P_6P_{14}P_4)}{(156)D_6} + V\beta_1\beta_6 \frac{(154)\ (P_{14}P_5P_4)}{(156)D_5}, \tag{141}$$

which is $V\beta_1\tau$. The scalar product of (141) and (140) must vanish. This verifies at once, by actual multiplication, the denominators being transformed as in (138), so that

$$\begin{aligned}D_5 &= (P_5P_{14}P_4) + c(315)\ (125)\ (123)^3\,(417),\\ D_6 &= (P_6P_{14}P_4) + c(316)\ (126)\ (123)^3\,(417).\end{aligned} \tag{142}$$

The determinant $S\pi\beta_1\tau$ then vanishes identically, and the test is completed.

Any irreducible quadratic vector, having β_1 a triple axis, and four other distinct axes, may be thrown into the form (136). For the only

limitation is that neither determinant (123) nor (156) shall vanish. We can evidently number the four single axes in pairs so that this restriction is not violated, for we may not have four distinct axes in one plane.

22. If, besides having β_1 as a triple axis, $F\rho$ has also a double axis, the number of possible ways of writing the vector is very large, whether we apply the method of limits, or introduce the vector $Q\rho$ as in the previous discussion of double axes. As the chief question considered in the present paper is the existence of the various types such that $F\rho$ is not reducible, it will be sufficient to note that we may write, for one triple and one double axis,

$$\beta_1 \frac{h(P_{25}P_6P\rho)}{(P_{25}P_6P_{14})} + \beta_2 \frac{h(614)\ (P_6P_{14}P\rho)}{(126)D_6} + \beta_6 \frac{h(124)\ (P_{14}P_{25}P\rho)}{(126)\ (P_{14}P_{25}[P_4+cP_{17}])} \tag{143}$$

obtained from (136) by putting $m\beta_2 + n\beta_5$ instead of β_5 and letting n approach zero. Similarly, for a vector with a triple axis and two double axes, we may take,

$$\beta_1 \frac{h(P_{25}P_{36}P\rho)}{(P_{25}P_{36}P_{14})} + \beta_2 \frac{h(314)\ (P_{36}P_{14}P\rho)}{(123)D_{36}} + \beta_2 \frac{h(124)\ (P_{14}P_{25}P\rho)}{(123)D_{25}}, \tag{144}$$

obtained from (143) by writing $m\beta_3 + n\beta_6$ instead of β_6 and letting n approach zero. In keeping with the notation already employed, we take $D_5\rho = 0$ as the polar plane with respect to the cone (137) of the vector β_5, i. e.

$$D_5\rho = (P_5\rho P_{14}P_4) + c[(315)\ (12\rho) + (31\rho)\ (125)]\ (123)^3\ (417)$$

with a similar meaning for $D_6\rho$. Whence we obtain D_{25} and D_{36} by writing β_2 and β_3, respectively, for ρ.

From their method of derivation, (143) and (144) are the most general vectors of their types. Moreover, (144) is always possible, because the only restriction is that the determinant (123) shall not vanish. But it was shown in Art. 15 that three multiple axes cannot lie in the same plane, i. e. (123) cannot vanish so long as $F\rho$ is irreducible.

The same cannot be said with regard to the above vector (143) because the determinants (123) and (126) must both be different

from zero. It may well happen that the triple axis, the double axis, and one of the single axes are in the same plane.

This special case, while it cannot be written in the form (143), is easily studied by the methods of Art. (19). Assume β_1 for the triple axis, β_2 for the double axis, β_3 and β_4 for the single axes. We cannot have both β_3 and β_4 coplanar with β_1 and β_2. Suppose β_4 coplanar with the two multiple axes. Then (123) cannot vanish if $F\rho$ is irreducible. Let the vector be thrown into the form (61). The conditions that β_4 be coplanar with β_1 and β_2 but distinct are given by (93). We have therefore case (c) of Art. 19. Moreover the condition that β_2 be a double axis becomes $A_{31} = 0$. This, together with (93) and (133) yield, as necessary and sufficient for the present case

$$\left.\begin{array}{l} A_{31} = A_{32} = A_{33} = A_{23}A_{12} - A_{13}A_{22} = 0;\ A_{11} \text{ and } A_{21} \text{ not} \\ \text{both zero; with } A_{22},\ A_{23},\ A_{12}, \text{ and } A_{13} \text{ all different from zero.} \end{array}\right\} \quad (146)$$

We thus have $\beta_4 = A_{13}\beta_1 + A_{23}\beta_2$, and may write

$$x_2x_3(A_{11}\beta_1 + A_{21}\beta_2) + (\beta_1 + b\beta_2)\,(A_{12}x_3x_1 + A_{13}x_1x_2), \quad (147)$$

(where b is a constant different from zero), as a normal form for a quadratic vector having a triple, a double, and a single axis in the same plane, and one other axis. As perhaps the simplest example of this case we may put $\beta_1 = i$, $\beta_2 = j$, $\beta_3 = k$, $A_{11} = 0$, and the remaining constants equal to unity. The equation (147) then becomes

$$i(zx + xy) + j(yz + zx + xy) \quad (148)$$

The cones (3) become

$$\left.\begin{array}{ll} \text{(a)} & zx(y + z) = 0, \\ \text{(b)} & z(yz + zx + xy) = 0, \\ \text{(c)} & x(zx + xy - y^2) = 0. \end{array}\right\} \quad (149)$$

The axes are (1, 0, 0), (0, 1, 0), (0, 0, 1) and (1, 1, 0). The quadrics $(yz + zx + xy) = 0$ and $(zx + xy - y^2) = 0$ pass through the vector (1, 0, 0) and have the same tangent plane there, viz. $y + z = 0$. Therefore (c) meets both (a) and (b) three times in the element i. This element is a double line for both (a) and (b). Hence it is a triple axis. At $j = \beta_2$, or (0, 1, 0), we have also a double line for (a) and for (b); giving a double axis. The axes i, j, and $i + j$ are co-

planar. The left members of (149) have no common factor. Thus all the conditions are satisfied for the case in question.

23. The case of a vector having two triple axes is of particular interest as affording the first example of a quadratic vector which cannot always be written in the form (61). For the triple axes and the single axis may be coplanar. To build up this type, we may write, as in (89),

$$Q(\rho) = i(12\rho)^2 + j(12\rho)\,(23\rho) + k(23\rho)\,(31\rho), \qquad (150)$$

giving β_2 a double axis, the tangent to (3) at β_2 being $(23\rho) = 0$. Put Q for P in (88), with $\alpha = \beta_4$. This gives, dropping the constant k_7,

$$\beta_4\frac{(567)(Q_5Q_6Q\rho)}{(456)(Q_5Q_6Q_7)} + \frac{\beta_5(567)(Q_5Q_6Q_{45})(Q_6Q_7Q\rho) + (647)(Q_5Q_6Q_7)(Q_5Q_6Q\rho)}{(456)(Q_5Q_6Q_7)^2} + \beta_6\frac{(457)(Q_{45}Q_5Q\rho)}{(456)\,(Q_{45}Q_5Q_7)}, \qquad (151)$$

which is of a type previously considered, viz. it is an example of the most general quadratic vector having two double axes. The tangent to (3) at β_5 is $(45\rho) = 0$. The single axes are β_1, β_6 and β_7. Let β_6 be replaced by $\beta_2 + t\beta_3 + at^2\beta_6$, and let t approach zero. Q_6 takes the form $Q_2 + tQ_{23} + t^2(Q_3 + aQ_{26})$ plus terms containing higher powers of t. But Q_2 and Q_{23} vanish identically. Also

$$Q_3 = i(123)^2, \qquad Q_{26} = k(123)\,(236), \qquad (152)$$

so that the determinants, (or scalar products), $(Q_5Q_3Q\rho)$ and $(Q_5Q_{26}Q\rho)$ give on expanding

$$(Q_5Q_3Q\rho) = (123)^2(235)\,(23\rho)\,(15\rho), \qquad (153)$$

$$(Q_5Q_{26}Q\rho) = (123)^2(236)\,(125)\,(12\rho)\,(52\rho) \qquad (154)$$

We have now merely to write $Q_3 + aQ_{26}$ instead of Q_6 in (151). The first term of (151) becomes

$$\beta_4\frac{(527)[(123)\,(235)\,(23\rho)\,(15\rho) + a(236)\,(125)\,(12\rho)\,(52\rho)]}{(452)[(123)\,(235)\,(237)\,(157) + a(236)\,(125)\,(127)\,(527)]}$$

which by a notation in keeping with that already used may be abbreviated

$$\beta_4 \frac{(527)\ E^5\rho}{(452)\ E^5{}_7}$$

where $E^5\rho$ denotes a quadratic form whose vanishing defines the cone through β_1 and β_5, having its curvature at β_2 determined by the constant a. We now have for the limiting value of (151),

$$\beta_4 \frac{(527)E^5\rho}{(452)\ E^5{}_7} + \beta_5 \frac{-(527)E^5{}_{45}E^7\rho + (247)E^5{}_7E^5\rho}{(452)\ (E^5{}_7)^2} + \beta_2 \frac{(457)(Q_{45}Q_5Q\rho)}{(452)\,(Q_{45}Q_5Q_7)}, \tag{155}$$

where, in the second term, $E^7\rho$ contains β_7 instead of β_5. This vector is also of a type previously examined, viz. it has one triple and one double axis. If the determinant $(527) = 0$ it becomes a binomial, in agreement with the vector (147). We have now to write $(\beta_5 + t\beta_4 + t^2b\beta_1)$ instead of β_1 and let t approach zero. When t approaches zero we have

$$Lim\ \frac{E\rho}{E_7} = \frac{(235)^2(23\rho)\ (45\rho) + a(236)\ (425)\ (52\rho)^2}{(235)^2(237)\ (457) + a(236)\ (425)\ (527)^2}, \tag{156}$$

the numerator being a quadric which vanishes on a cone through β_5 and β_2 with tangent planes at those elements respectively $(45\rho) = 0$ and $(23\rho) = 0$, and having the constant a arbitrary.

Again, $E^7\rho$ does not vanish in the limit, but becomes

$$(523)\ (237)\ (23\rho)\ (57\rho) + a(236)\ (527)\ (52\rho)\ (72\rho),$$

obtained by writing 7 for 5 and 5 for 1 in $E^5\rho$.

Now E_{45} may be expanded as

$$E_{45} = (123)\ (235)^2(154) + a(236)\ (125)^2(524),$$

and on writing for β_1 its new value we have terms containing the square and higher powers of t. We thus find

$$Lim\ \frac{E^5{}_{45}}{(E_7{}^5)} = \frac{b(523)\ (235)^2(154) + a(236)\ (425)^2\ (524)}{[(235)^2(237)\ (457) + a(236)\ (425)\ (527)^2]^2}$$

Considering finally the third term of (155), if we expand and simplify $(Q_{45}Q_5Q\rho)$ we have

$$(Q_{45}Q_5Q\rho) = (123)^2[(21\rho)\,(25\rho)\,(235)\,(154) - (23\rho)\,(15\rho)\,(125)\,(245)], \tag{157}$$

whence, writing for β_1 its value and taking the limit,

$$Lim\,\frac{(Q_{45}Q_5Q\rho)}{(Q_{45}Q_5Q_7)} = \frac{(23\rho)\,(45\rho)\,(245)^2 + b(25\rho)^2(235)\,(145)}{(237)\,(457)\,(245)^2 + b(257)^2(235)\,(145)}. \tag{158}$$

This result is, in form, like (156), but the constants a and b have different meaning; being, respectively, parameters whose vanishing causes the cones (3) to have inflectional elements at β_2 and β_5.

Collecting results, we have as the limiting form of (155),

$$\begin{aligned}
&\frac{\{\beta_4(527) + \beta_5(247)\}\{(235)^2(23\rho)\,45\rho) + a(236)\,(425)\,(52\rho)^2\}}{(235)^2(237)\,(457) + a(236)\,(425)\,(527)^2} \\
&\quad + \frac{\beta_2\{(23\rho)\,(45\rho)\,(245)^2 + b(25\rho)^2(235)\,(145)\}(457)}{(237)\,(457)\,(245)^2 + b(257)^3(235)\,(145)} \\
&\quad (527)\,\{b(523)\,(235)^2(154) + a(236)\,(425)^2(524)\} \\
&-\beta_5\,\frac{\{(523)\,(237)\,(23\rho)\,(57\rho) + a(236)\,(527)\,(52\rho)\,(72\rho)\}}{\{(235)^2(237)(457) + a(236)\,(425)\,(527)^2\}^2}
\end{aligned} \tag{159}$$

which is the most general form of quadratic vector with two triple axes. It is always possible; for the only limitation on the seven vectors is that the determinant (452) shall not vanish, it being always assumed that the vector is irreducible. Now we may choose either triple axis as β_5, with the tangent plane to (3) given by $(45\rho) = 0$. If this should pass through β_2 we have merely to choose β_2 instead of β_5; for the tangent at β_5 cannot also be tangent at β_2, the vector being irreducible.

If the three axes are coplanar, the determinant (257) vanishes, and the vector as above written becomes a binomial.

As a simple special case, let $i = \beta_2$, $j = \beta_5$, and $i + j = \beta_7$.

Take $k = \beta_3 = \beta_4$, making the tangent planes to (3) at i and j respectively $y = 0$ and $x = 0$. We cannot have both a and b zero, as this would give inflectional elements at both i and j and the vector would become reducible. Take $a = 0$, giving the cones (3) three common points in the plane $y = 0$ at i. Then β_6 disappears from the formula. Leaving b arbitrary, with $\beta_1 = i$, and noting that all the three-row determinants are either $+1$ or -1, we easily find

$$jxy - i\,(xy + bz^2), \tag{160}$$

as the value of (159). The cones (3) become

$$\left.\begin{array}{ll} (a)\ z(xy + bz^2) = 0, \\ (b)\qquad xyz = 0, \\ (c)\ y(x^2 - xy - bz^2) = 0. \end{array}\right\} \tag{161}$$

The tangent plane to the quadric cone whose equation is $xy + bz^2 = 0$ is, at the element i, the coördinate plane $y = 0$. The cones (a) and (b) both have i for a double element. Thus the cone (c), which at the element i consists of the plane sheet $y = 0$, has three coincident elements in common with the other cones (b) and (c). That is, i is a triple axis of (160).

At the element j, the quadric cones

$$xy + bz^2 = 0 \text{ and } x^2 - xy - bz^2 = 0$$

have the common tangent plane $x = 0$. The cones (a) and (b) have j for a double element. Thus the cone (c), which at the element j consists of the quadric sheet $x^2 - xy - bz^2 = 0$, touches one sheet of each of the other cones, and cuts one sheet, giving triple intersection, but the three consecutive elements are not coplanar. By change of coördinate planes, we can, if we wish, obtain non-degenerate cubics having the same order of contact at these elements. The remaining axis is the intersection of the plane $z = 0$ with the quadric cone $x^2 - xy - bz^2 = 0$, i. e. we have $x = y$.

As an example of greater generality, we may take a and b both arbitrary, with the axis β_7 not coplanar with either of the triple axes β_2 or β_5. Suppose $i = \beta_2$, and $j = \beta_5$ as before, and put $k = \beta_7$. Let the tangent planes to the cones (3) meet in the axis β_7, that is, $k = \beta_3 = \beta_4$. We may take β_1 any vector not coplanar with β_4 and β_5, (for we assume now b different from zero), and may put $\beta_1 = i$. We may take β_6 any vector not coplanar with β_2 and β_3, (for we assume a not zero), and may put $\beta_6 = j$. All the three-row determinants become $+1$, -1, or 0. The vector (159) becomes, aside from a constant factor,

$$j(a - b)yz + k(xy - az^2) \tag{162}$$

The cones (3) become

$$\left.\begin{array}{rl} y[(2a - b)z^2 - xy] &= 0, \\ x(xy - az^2) &= 0, \\ xyz &= 0. \end{array}\right\} \tag{163}$$

The existence of triple axes at i and at j may be verified as in the preceding example.

24. We may bring the discussion of triple axes into close relation with the theory of quadratic point-transformations by writing

$$R(\rho) = ix_2^2 + j(x_1x_2 - ax_3^2) + kx_2x_3 \qquad (164)$$

That $R(\rho)$ has β_1 for a triple axis independently of the value of the constant a may be verified by either of these three methods: by the rule of Art. 18; by forming the cones (3); or by noting that the scalar components of $R(\rho)$, as quadric cones, meet three times in β_1, so that R has β_1 for a triple zero. If, therefore, we write R for P in the general normal form (31) we shall have a general form for a quadratic vector with a triple axis and four other distinct axes. Similarly, by writing R for P in (88), we shall have one triple and one double axis. Various other forms with R in combination with preceding methods are evidently possible.

The vector $R(\rho)$ becomes reducible when β_1 is an element of inflection for the cones (3), as may be easily shown by applying the rule of Art. 18; a result we might anticipate, because three proper quadric cones cannot have three coincident elements in a plane. If the vector $F(\rho)$ has β_1 a triple axis, with β_1 an element of inflection for (3), we may use the preceding general methods.

25. As a comprehensive set of normal forms for irreducible quadratic vectors with a triple axis, we may select the four following,—

(a) If there is no double axis, (136) is always possible. For the only restrictions, other than irreducibility, are that the determinants (123) and (156) shall be different from zero. But the axes β_2, β_3, β_5, and β_6 are the four single axes. Since no four distinct axes are coplanar we may evidently so choose the numbers that these restrictions hold.

(b) If there is one double axis, use (143) provided neither single axis, β_3 nor β_6, is coplanar with the triple axis β_1, and the double axis β_2. If so, use (147). The generality of these forms has already been proved.

(c) If there are two double axes, use (144).

(d) If there are two triple axes, use (159).

26. When an axis is triple, but not of higher multiplicity, it has already been pointed out that at least one of the cones (3) has a

uniquely determined tangent plane at the triple axis. If, on the other hand, the axis is quadruple, we may distinguish two cases,—

1°. The polar vector of $V\rho F\rho$, viz. $V\rho' F\rho + V\rho\Phi(\rho, \rho')$, vanishes identically when the axis, (as β_1), is written for ρ; this is equivalent to the condition that all cones (3) have the axis for a double element,—which therefore counts for four intersections.

2°. The polar vector of $V\rho F\rho$ does not vanish identically when the axis is written for ρ.

27. Case 1° is easily disposed of, for the four elements of the determinant (63) vanish. Choose for β_2 and β_3 any two of the three single axes such that (123) does not vanish. The vector $F\rho$ may now be written, (after reducing β_1, β_2, and β_3 to zeros by a term $\rho S\delta\rho$),

$$\beta_1(A_{11}x_2x_3 + A_{12}x_3x_1 + A_{13}x_1x_2) + (\beta_2A_{21} + \beta_3A_{31})x_2x_3 \quad (165)$$

The remaining axis is

$$\beta_1A_{11}A_{21}A_{31} - (A_{12}A_{31} + A_{13}A_{21} - A_{21}A_{31})(\beta_2A_{21} + \beta_3A_{31}), \quad (166)$$

as may be directly verified. Conditions for coplanarity of this axis with a pair of the other axes are respectively, $A_{11} = 0$; $A_{21} = 0$; and $A_{31} = 0$. In these cases the fourth axis becomes, respectively, $\beta_2A_{21} + \beta_3A_{31}$; β_3; and β_2. Thus we cannot have two of the three single axes, (distinct), coplanar with the multiple axis, but the three may themselves be coplanar. In fact the conditions $A_{21} = 0$ or $A_{31} = 0$ agree with those already found that β_2 or β_3, respectively, may be double axes, for we may not have $A_{12} = 0$ nor yet $A_{13} = 0$ if the vector (165) is to be irreducible.

Without assuming the existence of three diplanar axes, we may throw the given vector into the form (92). If β_2 is a double axis we have as above $A_{31} = 0$, for, by inspection of the polar vector (96), this condition is independent of the terms in x_3^2. The vector now takes the form

$$\beta_1(A_{11}x_2x_3 + A_{12}x_3x_1 + A_{13}x_1x_2 + B_1x_3^2) + \beta_2(A_{21}x_2x_3 + B_2x_3^2) + \beta_3B_3x_3^2 \quad (167)$$

Let us now suppose β_3, till this time arbitrary, to lie in the tangent plane to the cones (3) at β_2. This is the same as requiring the tangent

plane to be $x_1 = 0$. By (96) this gives, (on putting $\rho = \beta_2$, $x_3 = x_1 = 0$),

$$A_{11} = 0.$$

If we next subtract from the vector the term

$$A_{21}x_3(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3)$$

which is of the form $\rho S\delta\rho$ and does not alter the axes, we shall remove the remaining term in x_2x_3. If the vector coefficient of c_3^2 be called ζ, the vector may be written

$$\beta_1[A_{13}x_1x_2 + (A_{12} - A_{21})x_3x_1] + \zeta x_3^2 \tag{168}$$

The polar vector of $V\rho F\rho$ now becomes

$$V\rho'\{\beta_1[A_{13}x_1x_2 + (A_{12} - A_{21})x_3x_1] + \zeta x_3^2\} + V\rho\beta_1A_{13}(x_1'x_2 + x_1x_2') + V\rho\beta_1(A_{12} - A_{21})(x_3'x_1 + x_3x_1') + 2V\rho\zeta x_3'x_3. \tag{169}$$

If the rule for a triple axis be applied to β_2, we shall now put β_2 for ρ' and β_3 for ρ, that is $x_1' = x_3' = 0$ and $x_1 = x_2 = 0$. This gives ζ parallel to the τ of the rule of Art. 18. Again, β_1 is parallel to the π of the same rule. Hence if β_2 is a triple axis, we must have $S\beta_2\beta_1\zeta = 0$. We may therefore take

$$\zeta = a_1\beta_1 + a_2\beta_2$$

and the vector becomes

$$\beta_1[A_{13}x_1x_2 + (A_{12} - A_{21})x_3x_1 + a_1x_3^2] + \beta_2a_2x_3^2, \tag{170}$$

which has only two distinct axes, β_1 and β_2, quadruple and triple. If a_1 is zero, β_2 is an inflectional element of the cones (3). We cannot have $a_2 = 0$ or $A_{13} = 0$ if the vector is to be irreducible.

It appears therefore that all possible quadratic vectors which are irreducible, and have β_1 for a quadruple axis of the sort where all cones (3) have double elements at β_1, are included under (165) and (170). It is so far assumed that β_1 is not of higher order than four.

28. For Case 2° we may follow a similar method. Take β_1 an axis, supposed at first to be at least triple. Take β_3 in the tangent plane to (3) at β_1. If β_1 is rendered a zero the vector may be written, β_2 being at present any vector such that (123) does not vanish,

$$\pi x_1 x_2 + \tau x_3^2 + \zeta x_2 x_3 + \mu x_2^2, \tag{171}$$

where the terms in $x_3 x_1$ have been removed as in the preceding article. By the rule of Art. 18, the condition that β_1 shall be a triple axis is $S\beta_1 \pi\tau = 0$, the vectors π and τ agreeing precisely with those of that article.

The condition that β_1 shall be a quadruple axis will appear as a relation between the vector coefficients in (171). As in Art. 17 we may write

$$\chi\epsilon = g\epsilon \tag{172}$$

If we take the space-derivative of both sides in the direction ϵ and then operate by $S\epsilon$, we have, remembering that any derivative of the unit-vector ϵ is at right angles to ϵ,

$$S\epsilon \frac{d}{dh_\epsilon} \chi\epsilon = -\frac{dg}{dh_\epsilon} \tag{173}$$

where the operation $\dfrac{d}{dh_\epsilon}$ means the same as $-S\epsilon\nabla$.

The condition for a quadruple axis is that the right member of (173), on writing β_1 for ρ after the differentiation, shall be independent of λ.

To obtain the condition in convenient form we have to expand the left side in terms of differential operations performed directly on the vector $V\rho F\rho$. We have, since $\chi\epsilon = \dfrac{d\nu}{dh_\epsilon}$,

$$S\epsilon \frac{d^2\nu}{dh_\epsilon^2} = S\epsilon \frac{d}{dh_\epsilon} \frac{d}{dh_\epsilon} \cdot \frac{\sigma}{T\sigma}, \text{ where } \nu = U\sigma = U\nabla S\lambda\rho F\rho, \text{ as in Art. 17,}$$

$$= S\epsilon \frac{d}{dh_\epsilon} \left\{ \frac{1}{T\sigma} \cdot \frac{d\sigma}{dh_\epsilon} - \frac{\sigma}{T^2\sigma} \cdot \frac{dT\sigma}{dh_\epsilon} \right\}$$

$$= S\epsilon \left\{ \frac{1}{T\sigma} \cdot \frac{d^2\sigma}{dh_\epsilon} - \frac{2}{T^2\sigma} \cdot \frac{d\sigma}{dh_\epsilon} \cdot \frac{dT\sigma}{dh_\epsilon} + \text{terms in } \sigma \right\},$$

by the ordinary rules for differentiation. If we now write $d\sigma = \phi d\rho$, so that $\dfrac{d\sigma}{dh_\epsilon} = \phi\epsilon$, and, as in the investigation of triple axes, write a

for a unit vector which is constant and coincides with ϵ when β_1 is written for ρ, the right member becomes, because $S\epsilon\sigma = 0$,

$$\frac{1}{T\sigma} \cdot \frac{d}{dh_\epsilon} \cdot S\alpha\phi\epsilon - \frac{2}{T^2\sigma} \cdot \frac{dT\sigma}{dh_\epsilon} \cdot S\alpha\phi\alpha \tag{174}$$

In the first of these two terms, we have to distribute the operator $\frac{d}{dh_\epsilon}$, differentiating first as if ϵ were constant, that is, equal to α, and second as if ϵ alone were variable. The two results are

$$\frac{1}{T\sigma} \cdot \frac{d}{dh_\alpha} \cdot S\alpha\phi\alpha + \frac{1}{T\sigma} \cdot S\alpha\phi \frac{d\epsilon}{dh_\epsilon}. \tag{175}$$

Since $S\alpha\phi\alpha$ is the same as the second derivative of $S\lambda\rho F\rho$ in the constant direction α, we have, when β_1 is put for ρ after the differentiation

$$\frac{1}{T\sigma} \cdot \frac{d}{dh_\alpha} S\alpha\phi\alpha = \frac{1}{T\sigma} \frac{d^3}{dh_\alpha{}^3} S\lambda\rho F\rho = \frac{1}{T\sigma} S\lambda \frac{d^3}{dh_\alpha{}^3} \rho F\rho \tag{176}$$

which is in the most convenient form for differentiation.

Taking next the vector $\frac{d\epsilon}{dh_\epsilon}$, occurring in the second term of (175), we note that the derivative of a unit vector is always perpendicular to the unit vector, hence

$$\frac{d\epsilon}{dh_\epsilon} = u\rho + v\nu, \tag{177}$$

where u and v are scalar coefficients; for ϵ, ν, and ρ form a rectangular system. Because σ is a homogeneous quadratic vector, $\phi\rho = 2\sigma$; and $S\alpha\sigma = 0$, hence the term in u disappears. The scalar v equals $-g$, for

$$\begin{aligned} \frac{d}{dh\epsilon} S\nu\epsilon &= 0 \\ &= S \frac{\epsilon d\nu}{dh_\epsilon} + S\nu \frac{d\epsilon}{dh_\epsilon}, \text{ by distributing,} \\ &= S\epsilon\chi\epsilon - v, \text{ by (177),} \\ &= -g - v, \text{ because } \chi\epsilon = g\epsilon, \text{ and } \epsilon^2 = -1. \end{aligned}$$

This result gives, for the second term of (175), the value

$$-\frac{gS\alpha\phi\nu}{T\sigma}. \tag{178}$$

The remaining term of (174) is similar to (178). For

$$g = -\frac{S\alpha\phi\alpha}{T\sigma}, \text{ identically, because } S\alpha\phi\alpha = S\lambda\frac{d^2}{dh_\alpha{}^2}\rho F\rho,$$

as in Art. 18. Again,

$$2T\sigma dT\sigma = -d(\sigma^2) = -2S\sigma d\sigma = -2S\sigma\phi d\rho = -2T\sigma S\nu\phi d\rho;$$

whence $\frac{dT\sigma}{dh_\epsilon} = -S\nu\phi\epsilon = -S\nu\phi\alpha = -S\alpha\phi\nu$, because ϕ is self-conjugate. Thus the two terms of (174) are together equal to

$$\frac{S\cdot\lambda}{T\sigma}\left\{\frac{d^3}{dh_\alpha{}^3} - 3g\frac{d^2}{dh_\nu dh_\alpha}\right\}\rho F\rho, \tag{179}$$

which is in convenient form for differentiation. If β_1 is a triple axis, g is independent of λ. If β_1 is to be a quadruple axis, it is necessary and sufficient that (179) shall be independent of λ. The reasoning by which this condition has been obtained is independent of the degree of the given vector $F\rho$, and applies, therefore, to vectors of any degree. The denominator $T\sigma$, as already shown, takes the form $S\lambda\beta\pi$, when β_1 is written for ρ. It can easily be shown that, for all homogeneous vectors, the expression (179) takes the form

$$\frac{S\lambda\beta\tau'}{S\lambda\beta\pi}, \tag{180}$$

where τ' is a new vector.[17] The condition that this fraction shall be independent of λ is $S\beta\pi\tau' = 0$.

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17 To prove this in general, we may write $F\rho = \alpha P + \beta Q + \gamma R$, where α, β, and γ form a rectangular unit system as in the text, and P, Q, R, are homogeneous scalar polynomials of degree n in x, y, z. If we write d and δ for two independent symbols of differentiation; and put $dP = P(\delta\rho)$, $\delta P = P(\delta\rho)$, and $d\delta P = \delta dP = P(d\rho, \delta\rho)$, it being understood that β is always put for ρ after the differentiation; with a similar notation for the first and second differentials of Q and of R; we must have $P\beta = Q\beta = R\beta = 0$, because β is a zero of the

29. Applying this condition for a quadruple axis to (171), the third derivative of $\rho F\rho$ in the direction α is $3\alpha F\alpha$, because $\rho F\rho$ is homogeneous of degree 3. The other differential operation is the same as $S\nu\nabla\cdot S\alpha\nabla$. We may write β_3 for α and $\beta_3\beta_1$ for ν, and may assume these vectors to form a rectangular unit system, although the homogeneity of the conditions makes this assumption unnecessary. The operation $S\beta_3\beta_1\nabla\cdot S\beta_3\nabla\cdot$ on $V\rho F\rho$, where $F\rho$ has the form (171), yields, (forming the second differential vector and writing β_3 for $d\rho$ and $\beta_3\beta_1$ for $\delta\rho$ and β_1 for ρ),

$$V\beta_3\pi + V\beta_1\zeta,$$

and $V\alpha F\alpha$ gives $3V\beta_3\tau$. By writing $\tau = w\beta_1 + g\pi$, and substituting results in (179), we find as the condition that β_1 shall be a quadruple axis,

$$S\beta_1\pi(w\beta_3 + g\zeta) = 0. \tag{181}$$

30. It will now be most convenient to distinguish two subcases, according as g is, or is not, zero. If not, we may add to the vector $F\rho$ a term $\frac{wx_2\rho}{g}$ which will remove from the vector π its β_1 component, leaving it parallel to τ. We then have $F\rho$ as

$$\pi(x_1x_2 + gx_3^2) + \zeta x_2x_3 + \mu x_2^2 \tag{182}$$

The condition for a quadruple axis then appears as

$$S\beta_1\pi\zeta = 0. \tag{183}$$

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given homogeneous vector $F\rho$. Also, because γ is the normal to the cones (3), $P(\alpha) = R(\alpha) = 0$. Applying the rule of Art. 18 we find

$$\pi = \alpha P(\gamma) + \gamma R(\gamma), \quad \text{and} \quad \tau = \alpha P(\alpha, \alpha) + R(\alpha, \alpha) - 2\alpha Q(\alpha).$$

From the relations $S\beta\pi\tau = 0$ and $g = \frac{S\lambda\beta\tau}{S\lambda\beta\pi}$, we have $\tau = u\beta + g\pi$, where u is a scalar and g has the same meaning as in the text. Operating by $S\cdot\alpha\beta$ gives

$$g = \frac{S\alpha\beta\tau}{S\alpha\beta\pi} = \frac{R(\alpha, \alpha)}{R(\gamma)}$$

for all homgeneous vector-polynomials, β being a triple axis. If we write $F(\alpha, \alpha, \alpha)$ for the third derivative of F in the direction α, and perform the indicated operations, we find all terms not of the required form cancel, and

$$\tau' = F(\alpha, \alpha, \alpha) - 3\alpha Q(\alpha, \alpha) - 3g\{\alpha P(\gamma, \alpha) + \gamma R(\gamma, \alpha) - \alpha Q(\gamma) - \gamma Q(\alpha)\}.$$

We may therefore write

$$\zeta = u\beta_1 + g_1\pi, \tag{184}$$

and $F\rho$ becomes

$$\pi(x_1x_2 + gx_3^2 + g_1x_2x_3) + u\beta_1x_2x_3 + \mu x_2^2, \tag{185}$$

the most general quadratic vector having β_1 a quadruple axis, a determinate tangent plane to the cones (3) at β_1, and β_1 not an inflectional element of these cones. It is evident that if π lies in the tangent plane it is an axis and, conversely, if an axis other than β_1 exists in the tangent plane, this axis coincides with π. Let us suppose, as a very special case, that π is a double or triple axis, which we may take as β_3, (since β_3 is any vector in the tangent plane). The tangent plane to (3) at β_3 must be distinct from the tangent plane at β_1 since cubics have no double tangent, and $F\rho$ is assumed not reducible. Applying these conditions according to the methods already exemplified, we have, as the polar vector of $V\rho F\rho$ at β_3,

$$V\beta_3\{u\beta_1x'_2 - gx_1'\beta_1 - gx_2'\beta_2\}$$

which obviously cannot determine, by its vanishing, a unique tangent plane at β_3. Thus the supposed case is impossible; and π cannot be a double axis.

This possibility disposed of, the vector (185) must always have at least two axes not in the plane $(31\rho) = 0$. Let β_2 be one of these. The vector becomes

$$\pi(x_1x_2 + gx_3^2 + g_1x_2x_3) + u\beta_1x_2x_3 + a\beta_2x_2^2, \tag{186}$$

where a is a scalar constant. This form, therefore, is equally general with (185), β_1 being of order not higher than four.

If we consider the vector π, and the scalars u and a, as determined by assigned axes β_4 and β_5, I shall now show that the determination is uniquely possible, aside, obviously, from a scalar factor. If we write

$$x_1x_2 + gx_3^2 + g_1x_2x_3 = C\rho$$

the conditions that $F\beta_4$ and $F\beta_5$ shall be parallel, respectively, to β_4, and β_5 may be written

$$\begin{aligned} V\beta_4[C_4\pi + u\beta_1(314)\,(124) + a\beta_2(314)^2] &= 0 \\ V\beta_5[C_5\pi + u\beta_1(315)\,(125) + a\beta_2(315)^2] &= 0, \end{aligned} \tag{187}$$

two vector equations equivalent, in general, to four scalar equations. Multiplying the first by β_5 and the second by β_4 and taking scalars we obtain two equations,

$$\begin{aligned}(45\pi)C_4 + (451)\,(314)\,(124)u + (452)\,(314)^2a = 0\\ (45\pi)C_5 + (451)\,(315)\,(125)u + (452)\,(315)^2a = 0,\end{aligned} \tag{188}$$

homogeneous in the three unknowns (45π), u, and a. The two-row determinants from the coefficients cannot all vanish if $F\rho$ is not reducible. For the determinant of the second and third columns is

$$(451)\,(452)\,(314)\,(315)\,\{\,(124)\,315) - (125)\,(314)\,\}$$

which by a transformation already used becomes the product of determinants

$$-\,(451)^2(452)\,(314)\,(315)\,(123). \tag{189}$$

Considering these factors in order, if $(451) = 0$, the axes β_1, β_4, and β_5 are coplanar, and we may put

$$\beta_5 = m\beta_1 + n\beta_5.$$

The matrix of the coefficients becomes

$$\begin{matrix} C_4, & 0, & m(412)\,(314)^2,\\ mn(231)\,(314) + n^2C_4, & 0, & mn^2\,(412)\,(314)^2,\end{matrix}$$

The axes β_4 and β_5 being assumed distinct, neither m nor n is zero. We cannot have $(412) = 0$ for the four axes β_1, β_2, β_4, β_5, would be coplanar and $F\rho$ would be reducible. We cannot have $(314) = 0$ for we cannot have two distinct axes in the tangent plane to (3), viz. $(31\rho) = 0$. And we have (123) different from zero by hypothesis. Hence this matrix cannot have its rank reduced to one.

Taking the second factor of (189), if $(452) = 0$, the three single axes are coplanar. We may put

$$\beta_5 = m\beta_2 + n\beta_4.$$

The matrix of the coefficients becomes

$$\begin{matrix} C_4, & m(421)(314)\,(124), & 0,\\ mn(312)\,\{(234) + g_1\,(124)\} + n^2C_4, & m(421)\,\{m(312) + n(314)\}n(214), & 0,\end{matrix}$$

and the only non-vanishing determinant is seen to equal

$$-\,gm^2n(124)^4(123)$$

no factor of which can vanish under the hypotheses.

Taking the third factor of (189), if $(314) = 0$, β_4 lies in the tangent plane. The two non-vanishing determinants of the matrix reduce to

$$g(124)^2(452)\,(315)^2 \qquad \text{and} \qquad g(124)^2(451)\,(315)\,(125).$$

By hypothesis, g is not zero. We cannot have $(124) = 0$ since β_2 was taken without the tangent plane. We cannot have $(315) = 0$, since there cannot be two axes in the tangent plane, other than β_1. Similarly, we cannot have $(451) = 0$. Hence if both determinants vanish, we have simultaneously $(452) = 0$ and $(125) = 0$, the four axes β_1, β_2, β_4, and β_5, coplanar, which is impossible under the hypotheses.

Taking the fourth factor of (189), the same reasoning holds. The last factor differs from zero by hypothesis. Hence the solution of (188) is unique, aside from a factor of proportionality. This factor aside, we find for the constant a the value, by an easy computation

$$a = -\,(123)\,(145)\,\{\,(314)\;(315)\,(245) + g(124)\,(125)(145)\},\qquad (190)$$

the result being independent of g_1. For the constant u we find

$$u = (123)\,(245)\,[-\,(314)\,(315)\,(345) + g(145)\,\{\,(314)\,(125) + (315)\,(124)\} + g_1(145)\,(314)\,(315)].\qquad (191)$$

In determining the vector π we shall not fail to remark that neither C_4 nor C_5 can be zero. This can be shown from the fact that, if C_4 is zero, (for example), $F\rho$ is a limiting case of a reducible vector, $C\rho$ defining by its vanishing a quadric having four elements in common with (3) at β_1 and passing through β_2, so that if it passes through another axis we have six on a quadric. Better, if $C_4 = 0$, it is evident from (186) that β_4 lies in the plane of β_1 and β_2. But x_3 vanishes when β_4 is put for ρ. Hence β_4 must coincide with β_2 in direction, contrary to hypothesis. Similarly, C_5 cannot vanish.

The most natural way to determine π, provided (145) does not vanish, is by means of the identity

$$\pi(145) = \beta_1(45\pi) + \beta_4(51\pi) + \beta_5(14\pi)\qquad (192)$$

The component (45π) is given by (189). Multiplying both equations of (187) by β_1 and taking scalars we have

$$\begin{aligned} C_4(14\pi) + a(142)\,(314)^2 &= 0,\\ C_5(51\pi) + a(512)\,(315)^2 &= 0. \end{aligned}\qquad (193)$$

These equations complete the solution. Similarly, if (245) does not vanish we can express π in terms of β_2, β_4, and β_5.

31. The remaining subcase supposes β_1 an inflectional element of the cones (3), so that $g = 0$. This is equivalent to saying that the vector τ of (171) shall be parallel to β_1. We then have, as the condition that β_1 shall be a quadruple axis, by (181),

$$S\beta_1\pi\beta_3 = 0, \tag{194}$$

for the scalar u cannot vanish if $F\rho$ is not reducible. (If so, we should have x_2 a factor of (171)). The condition (194) requires that π shall lie in the tangent plane $x_2 = 0$ to the cones (3) at β_1. Since β_3 is so far any vector in that plane, we may take $a\beta_3 = \pi$, and (171) becomes

$$a\beta_3x_1x_2 + u\beta_1x_3^2 + \zeta x_2x_3 + ux_2^2 \tag{195}$$

Since we cannot now have any axis in the tangent plane distinct from β_1, (for if so we should have four axes in a plane), we may assume β_2 an axis, and, by a properly chosen term in ρ, remove the term in x_2^2 from the vector $F\rho$. We then have

$$a\beta_3x_1x_2 + u\beta_1x_3^2 + \zeta x_2x_3, \tag{196}$$

as the most general form for this sub-type. (The constants a and u are, of course, altered by addition of a term in ρ). We have the vector ζ and the scalars a and u at our disposal to determine two more axes β_4 and β_5 as in the former subcase. This, however, presents no new difficulty.

32. The methods already exemplified are amply sufficient to impose on the two vectors (186) or (196) conditions that β_2 shall be a double or a triple axis. As they are the most general possible vectors of their types, as was shown, they contain all further special cases, having β_1 for a quadruple axis.

As an example, let $\beta_1 = i$, $\beta_3 = j$, $\beta_3 = \pi = k$, and let a third single axis be $j + k$, so that the three single axes are coplanar, and one of them is in the tangent plane, $(y = 0)$, to (3) at i. Let $g = 1$ and $g_1 = 0$. We find from (186)

$$k(xy + z^2) + jy^2$$

as the value of $F\rho$, and the cones (3) become

$$y(xy + z^2 - yz) = 0, \quad xy^2 = 0, \quad x(xy + z^2) = 0.$$

Any linear function of the left members gives a uniquely determined tangent plane $y = 0$ at the element i, provided that the coefficient of the first is not zero. That j, k, and $j + k$ are axes verifies by direct substitution, as also the quadruple character of the axis i.

As a second example, let i be a quadruple, and j a triple axis. Let the tangent plane at i be $y = 0$ and at j be $z = 0$. Let $g = 1$ and $g_1 = 0$. The vector

$$j(xy + z^2) + iyz$$

satisfies these conditions, the cones (3) becoming

$$z(xy + z^2) = 0, \; yz^2 = 0, \text{ and } x^2y + xz^2 - y^2z = 0.$$

That there are no axes except i and j is obvious by inspection, and the quadruple character of i verifies easily. This is a special case of (186).

The constant g_1 is closely related to the aberrancy [18] of any cubic curve obtained from the cones (3) by plane sections through β_1, and vanishes if the aberrancy of such a cubic vanishes at β_1.

Conditions that a quadratic vector may have an axis of order higher than four may be obtained by similar methods, but such conditions are not needed for the complete determination of the axes of a quadratic vector. For suppose a vector to have an axis of the fifth order. It is evident that there can be, at most, two other axes if the vector is irreducible. An examination of these is sufficient proof of the quintuple character of the multiple axis. Or again, suppose a vector to have only two axes. If these be tested, by the rules above given, one will be found to be of the fourth order, at least. It will then be of the fifth, or of the sixth, order, according as the remaining axis is of order two or one.

If a quadratic vector has only one axis, that must be of the seventh order. As a simple example,

$$i(xy + z^2 + y^2) + jy^2$$

has i for its only axis, the cones (3) becoming

$$y^2z = 0, \quad z(xy + y^2 + z^2) = 0, \quad y(y^2 + z^2) = 0.$$

The tangent plane at i is not determinate, all combinations of these equations having a double element at i. This is, therefore, not a special case of (186).

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18 Salmon, Higher Plane Curves, 3rd. Ed. Art. 407.

PART FOUR.

33. Our analysis of quadratic vectors is now complete, in the sense that the nature of any given vector can be completely determined by the foregoing methods. Furthermore, normal forms, or model vectors, have been given including all possible vectors. I shall apply what precedes to the study of a general, and very simple, normal form, by means of which the properties of a quadratic vector are made to depend on those of a pair of vectors of the first degree; namely

$$V\phi\rho\theta\rho + \rho S\delta\rho, \tag{197}$$

where ϕ and θ are linear in ρ, and δ is a constant vector.[19] The first term of this vector, the vector product $V\phi\rho\theta\rho$, has three zeros, for there exist three directions, in general distinct, which are altered in the same manner by the operations ϕ and θ. These directions [20] are the axes of the linear vector function $\phi^{-1}\theta$. Let three vectors along these directions be β_1, β_2, and β_3. Let them be converted by ϕ into λ_1, λ_2, λ_3, respectively. We may then write

$$\phi\rho\cdot(123) = \lambda_1(23\rho) + \lambda_2(31\rho) + \lambda_3(12\rho),$$

or with the notation already adopted,

$$\phi\rho = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 \tag{198}$$

and also

$$\theta\rho = g_1\lambda_1 x_1 + g_2\lambda_2 x_2 + g_3\lambda_3 x_3, \tag{199}$$

where g_1, g_2, and g_3 are the roots of the cubic in $\phi^{-1}\theta$, that is they satisfy three relations of the form $\phi^{-1}\theta\beta = g\beta$. Whence it follows that $\theta\beta = g\phi\beta$ and $V\phi\beta\theta\beta = 0$., for β_1, β_2, or β_3. If we now multiply together the corresponding members of (198) and (199), introducing the notation

19 In a former paper, (Phil. Mag. Jan. 1909, page 124) I gave this form (without proof), in connection with differential operators of the second order, the symbol ∇ being written for ρ).

20 Cf. the appendix by the late Prof. C. J. Joly to Hamilton's 'Elements of Quaternions,' 2nd. Ed. p. 363.

$(g_3 - g_2)V\beta_2\beta_3 = \alpha_1$, $(g_1 - g_3)V\beta_3\beta_1 = \alpha_2$, $(g_2 - g_1)V\beta_1\beta_2 = \alpha_3$, (200)
we find identically

$$V\phi\rho\theta\rho = \alpha_1 x_2 x_3 + \alpha_2 x_3 x_1 + \alpha_3 x_1 x_2. \tag{201}$$

It has already been shown that any quadratic vector of the general type, or any having three distinct diplanar axes, can be thrown into the form of the right member, by means of the addition of a properly chosen term $\rho S\delta\rho$. We have now to examine the converse of the process by which (201) was just obtained, viz. to convert the right member into the left, or to factor, vectorially, into the linear vectors $\phi\rho$ and $\theta\rho$. In the most general form of quadratic vector this will be possible. For we may write

$$\begin{aligned}\phi\rho &= h_1 x_1 V\alpha_2\alpha_3 + h_2 x_2 V\alpha_3\alpha_1 + h_3 x_3 V\alpha_1\alpha_2,\\ \theta\rho &= c_1 x_1 V\alpha_2\alpha_3 + c_2 x_2 V\alpha_3\alpha_1 + c_3 x_3 V\alpha_1\alpha_2,\end{aligned}$$

where the six constants h and c are undetermined. If we take the vector product of corresponding members, utilizing the identity, proved in all works on vectorial algebra,

$$V\cdot V\alpha_1\alpha_2 V\alpha_3\alpha_1 = \alpha_1 S\alpha_1\alpha_2\alpha_3$$

with two others of like form, and compare with (201), we find these three relations to determine the six constants,

$$h_3c_2 - h_2c_3 = h_1c_3 - h_3c_1 = h_2c_1 - h_1c_2 = \frac{1}{S\alpha_1\alpha_2\alpha_3} \tag{202}$$

whence, evidently, there are an infinite number of ways to write down $\phi\rho$ and $\theta\rho$. For example, a simple, although unsymmetrical, solution, is

$$h_1 = 1, \quad h_2 = -1, \quad h_3 = 0, \quad c_1 S(\alpha_1\alpha_2\alpha_3) = -1, \quad c_2 = 0, \quad c_3 S(\alpha_1\alpha_2\alpha_3) = +1.$$

If we let the resulting values of ϕ and θ, or any two we may construct satisfying the conditions, be called ϕ_0 and θ_0, then the new pair,

$$\phi = u\phi_0 + v\theta_0, \quad \theta = u_1\phi_0 + v_1\theta_0,$$

will also satisfy them, provided $uv_1 - u_1v = 1$. It thus appears that we have $V\phi\rho\theta\rho$ determinable by fifteen scalars; each linear vector in general involving nine scalars, but in the vector product we have the four parameters u, v, u_1 v_1, with the restriction as given.

This process assumes that $V\alpha_2\alpha_3$, $V\alpha_3\alpha_1$, and $V\alpha_1\alpha_2$ are diplanar, since $\phi\rho$ and $\theta\rho$ were assumed in terms of them. This, in turn, implies that α_1, α_2, and α_3 are diplanar, in general true, but not necessary, even in what I have called a vector of the first, or general type. For it was shown that if $F\rho$ has three zeros, and if, of the four remaining axes, three are coplanar, the last or seventh axis will also be a zero, (cf. Art. 6), and the vector itself will be always in one plane, i. e. α_1, α_2, α_3 will be coplanar.

The possibility of the normal form (197) therefore depends, when sets of coplanar axes exist, upon so selecting the zeros that, of the four other axes, no three shall be coplanar, while the three zeros themselves cannot be coplanar. Each relation of coplanarity between sets of three vectors diminishes the number of possibilities. Thus if no three axes are coplanar, three can be chosen in thirty-five ways, each leading to a separate form like the right member of (201). If three axes are coplanar, the number of possibilities reduces to thirty-one, since we must now exclude any choice of three zeros from the four non-coplanar axes, and three can be chosen from four in four ways.

That the choice of three diplanar axes, so that of the four that remain no three shall be coplanar, will always be possible may be proved by exhaustion as follows. Let β_1, β_2, and β_3 be chosen as any three diplanar axes. Let β_4, β_5, and β_6 be any remaining three which are not coplanar. If β_7 is coplanar with two of this latter three, suppose $(567) = 0$. If of the remaining four vectors β_1, β_2, β_3, β_4, no three are coplanar, the problem is solved, since (456) is not zero.

Of these four, (123) does not vanish by hypothesis. If β_7 is coplanar with two out of the three β_1, β_2, β_3, suppose $(237) = 0$. Then (127) is not zero, since no four axes are coplanar.

Choose β_1, β_2, and β_7 to be zeros. If of the four others β_3, β_4, β_5, β_6, no three are coplanar, the problem is solved. Of the four, (456) is not zero by hypothesis, and (356) is not zero for $(567) = 0$ and no four axes are coplanar. If β_3 and β_4 are coplanar with either β_5 or β_6 (so far treated alike), suppose $(345) = 0$.

Choose β_3, β_5, and β_6 to be zeros. If of the four others β_1, β_2, β_4, β_7, no three are coplanar, the problem is solved. Of these four, (127) and (247) can neither be zero, since $(237) = 0$. Now (124) and (147) cannot both be zero. Two cases are to be distinguished,—

Case 1. $(124) = 0$. Choose β_1, β_4, and β_7 to be zeros. Of the four others, β_2, β_3, β_5, β_6, (235) and (236) are not zero since $(237) = 0$. And (256) and (356) are not zero, since $(567) = 0$. The problem is therefore solved. We may still have $(163) = 0$, when the seven axes

will lie in sets of three on five planes, no three planes having a common axis, except those through β_3.

Case 2. $(147) = 0$. If β_1, β_3, and β_6 are diplanar, choose them to be zeros. Of the four others, β_2, β_4, β_5, β_7, (247) and (257) cannot be zero, since $(237) = 0$. (457) cannot be zero since $(147) = 0$. (245) cannot be zero since $(345) = 0$. The problem is then solved. But if $(136) = 0$ choose β_2, β_4 and β_6 to be zeros, if diplanar. Then of the four others β_1, β_3, β_5, β_7, we cannot have (135) nor (137) zero because $(136) = 0$. We cannot have (157) nor (357) zero since $(567) = 0$. The problem is then solved. But if $(246) = 0$, choose β_1, β_2, and β_5 to be zeros. Of the four others, β_3, β_4, β_6, β_7, we cannot have (346) nor (467) zero, since $(246) = 0$. We cannot have $(347) = 0$ nor $(367) = 0$ since $(237) = 0$. The problem is then solved if (125) does not vanish. But the three vectors β_1, β_2, β_5 are the intersections of planes through the other four axes taken in pairs. It has just been shown that no three of these four can be coplanar. Therefore (125) cannot vanish, and the proof is complete. The axes now lie in sets of three on six planes, the four not chosen as zeros being at the intersection of three planes each.

The form (197) is then always possible for a quadratic vector of the first or general class, having just seven distinct axes.

34. If an irreducible quadratic vector possesses one or more double axes, but no triple axis, it may still be thrown into the form (197), but the manner of obtaining a proper ϕ and θ is somewhat different. Let, at first, each double axis be replaced by two single axes, one coinciding with the original double axis, the other lying in the original tangent plane to the cones (3) at the double axis. Then let the selection of three axes proceed as in Art. 33, these three to be diplanar, and of the four others no three to be coplanar. Then let the axes in the respective tangent planes approach their original positions. We now distinguish two cases.

Case 1. If the three axes selected remain single axes, let them be reduced to zeros, the vector then taking the form of the right member of (201). The vectors α_1, α_2, α_3, cannot be coplanar, and the factorization into the form (197) proceeds as before.

Proof. Suppose α_1, α_2, and α_3 to be coplanar. Put

$$\alpha_3 = m\alpha_1 + n\alpha_2.$$

The vector $F\rho$ takes the form

$$\alpha_1(x_2x_3 + mx_1x_2) + \alpha_2(x_3x_1 + nx_1x_2).$$

Four of the seven axes are zeros, viz. the four intersections of

$$x_2x_3 + mx_1x_2 = 0. \quad x_3x_1 + nx_1x_2 = 0.$$

There can be no other axis without the plane of a_1 and a_2. For the above pair of simultaneous equations cannot have five solutions since the number is finite. No axis not given by a solution of these equations can be a zero. That is, the remaining axes are not zeros. Hence they are in the plane of a_1 and a_2. Since, by hypothesis for this case, the three selected zeros are single axes, the fourth zero is single. Of the remaining non-zero axes, one is therefore a double axis. In virtue of the method by which the first three zeros were selected, the last non-zero axis cannot lie in the tangent plane at the double axis, which, therefore, cannot be the plane of a_1 and a_2.

Consider the normal form (88); which is precisely the present case,—three single axes being zeros, the double axis being β_5, and the tangent plane at β_5 passing through a, which by hypothesis, is distinct from β_5. If (88) lies in a constant plane, that plane is therefore the tangent plane, contrary to the above result. Hence the vector cannot lie in a constant plane; that is, a_1, a_2, a_3, cannot be coplanar.

Case 2. If the three vectors selected do not remain single in the limiting vector, let the double axis be β_1 and the other be β_2 and let the vector be thrown into the form (171), and β_2 be rendered a zero. We then have

$$F\rho = \pi x_1x_2 + \tau x_3^2 + \zeta x_2x_3. \tag{203}$$

The vectors π, τ, and ζ will not be coplanar.

Proof. Suppose π, τ, ζ, to be coplanar. Put

$$\zeta = m\pi + n\tau.$$

The vector $F\rho$ takes the form

$$\pi(x_1x_2 + mx_2x_3) + \tau(x_3^2 + nx_2x_3).$$

By the same reasoning as in Case 1, the non-zero axes must lie in the plane of π and τ. By virtue of the method of selecting the first three zeros, if the non-zero axes are three single axes, they are not coplanar. Hence they cannot be all single. One non-zero axis is therefore double. By the same reasoning as in Case 1, the tangent plane at this latter double axis cannot be that of π, and τ.

Consider the model form obtained by writing Q for P in (88), which is precisely the present case. As above, the vector, if in a

constant plane, is in the tangent plane at β_5, in contradiction with the result just obtained. Hence π, τ, and ζ are not coplanar.

I shall now show that the right member of (203) can always be thrown into the form $V\phi\rho\theta\rho$ when ζ, π, and τ are not coplanar. Write

$$\phi\rho = x_1 V\pi\tau + x_2 V\zeta\pi + x_3\mu, \tag{204}$$

where μ is to be determined. Also

$$\theta\rho = g_1 x_1 V\pi\tau + g_2 x_2 V\zeta\pi + (g_1\mu + g_3 V\pi\tau)x_3, \tag{205}$$

these simultaneous forms of ϕ and θ corresponding to coincidence of two axes of the linear vector $\phi^{-1}\theta\rho$. If we now take the vector product $V\phi\rho\theta\rho$ and equate to (203) we find, comparing vector coefficients of x_1x_2, x_3^2, and x_2x_3,

$$(g_2 - g_1)S\zeta\pi\tau = 1, \quad g_3 S\pi\mu = 1, \quad S\tau\mu = 0,$$
$$\zeta = (g_1 - g_2)VV\zeta\pi\cdot\mu - g_3\pi S\zeta\pi\tau.$$

From these equations we obtain the solution, for g_3 and μ,

$$g_3 S\zeta\pi\tau = -1, \qquad \mu = V\zeta\tau + V\tau\pi \tag{206}$$

whence ϕ and θ are known, g_1 and g_2 being arbitrary provided their difference be constant.

35. Taking next the case of a quadratic vector having a triple axis, but no axis of higher order, a number of situations may arise. Suppose first that we have one triple and four single axes. Let the triple axis be β_1. It is always possible so to choose β_2 and β_3 that of the two other axes neither shall lie in the tangent plane to (3) at β_1, (because only one single axis can lie in that plane), and, at the same time, so that these two remaining axes shall not be coplanar with β_1, (because four axes cannot be coplanar). Let β_1, β_2, and β_3 be rendered zeros. We then have the form of the right member of (201). The three vectors α_1, α_2, and α_3 must be diplanar, and we can factor into $V\phi\rho\theta\rho$ as before.

Proof. Suppose $(\alpha_1\alpha_2\alpha_3) = 0$. These three vectors will not all be parallel, for $F\rho$ is not reducible. Let one of them be expressed in terms of the other two, (non-parallel). The scalar coefficients of these two define, by their vanishing, two quadrics. Two cases may arise.

Case 1. These quadrics are tangent at β_1. Neither of the single axes not first chosen to be zeros can be now a zero. But all non-zero

axes are in the plane of the vectors α_1, α_2, α_3. But β_1 must lie in that plane by reason of its triple character. This is contrary to the hypothesis that β_1 and the two axes are diplanar.

Case 2. The quadrics are not tangent at β_1. One of the two single axes not selected to be zeros must be a zero, since the quadrics have four intersections. But the plane of the vectors α contains three axes, is therefore the tangent plane to (3) at β_1, and passes through the non-zero axis, contrary to hypothesis.

A similar proof can be obtained analytically by considering the condition of Art. 19 that β_1 shall be a triple axis.

Under case 2 use was made of the fact that, if the plane of a quadratic vector is constant, and if two of the three axes which it must then possess in that plane are coincident, the plane is tangent to the cones (3) at the double axis,— assuming this tangent plane to be uniquely determined. An analytical proof of this is desirable. Suppose first that the double axis is a zero, as in the case of the triple axis just considered. Let coördinate axes be taken so that i is the zero, j also in the constant plane, so that the vector becomes

$$i(bxy + cy^2 + \text{terms in } z) + j(b_1xy + c_1y^2 + \text{terms in } z).$$

The equation to determine axes in the plane $z = 0$ then becomes

$$x(b_1xy + c_1y^2) = y(bxy + cy^2).$$

That there may be only one axis other than $y = 0$ we must have $b_1 = 0$. But the polar vector of $V\rho F\rho$ at i takes the form

$$k(b_1y' + \text{a term in } z').$$

Hence the condition that the tangent plane to (3) shall be the plane $z = 0$ is also that we have $b_1 = 0$.

If the double axis is not a zero, we shall have the vector in the form

$$i(ax^2 + bxy + cy^2 + \text{terms in } z) + j(b_1xy + c_1y^2 + \text{terms in } z),$$

where i is taken as an axis. The equation to determine axes in the plane $z = 0$ becomes

$$x(b_1xy + c_1y^2) = y(ax^2 + bxy + cy^2)$$

giving $b_1 = a$ if there is to be but one axis other than i. But the polar vector becomes at i,

$$V(ix' + jy' + kz')ia + Vi(jb_1y' + \text{a term in } z').$$

The only terms other than those in z' are

$$-kay' + kb_1y'$$

Hence $b_1 = a$ gives $z = 0$ as the tangent plane to (3) at i.

The reasoning of the preceding article may now be extended to cover the case of a quadratic vector with one triple axis and either one or two double axes. We proceed as in Art. 34. If the axes originally selected to be zeros along with β_1 remain single axes, we still obtain the form of (201). The resulting vectors α cannot be coplanar and we may factor as before. If, on the other hand, the axes selected to be zeros approach each other, we obtain the form of (203). The reasoning already employed shows that the plane of the resulting vector cannot be constant, and we may factor as in Art. 34.

36. If the given quadratic vector has two triple axes and one single axis, the preceding reasoning fails. A normal form for this case is (159). It will be simplest to consider two subcases, according as the three distinct axes are, or are not, diplanar.

Case 1. The three axes β_2, β_5, and β_7 are diplanar. Let these axes be made zeros by adding to the normal form (159) a proper term of the form $\rho S\delta\rho$. The vector then takes the form of the right member of (201). The constant vectors α will not be in the same plane, and we factor into $V\phi\rho\theta\rho$ as in Art. 33.

Proof. Suppose the vectors α to be coplanar. Let one of them be expressed in terms of the other two. The scalar coefficients of these two will then define, by their vanishing, two quadrics. The four intersections of these quadrics can only be at β_2, β_5 and β_7, since there are no other axes. Hence one of the axes β_2 or β_5 must be a double intersection for the two quadrics. Suppose the axes to be numbered so that β_5 is a double intersection for the two quadrics. The four intersections of these quadrics count as four of the seven axes of the quadratic vector. The plane of the vectors α must contain the other three, which, however, can only be at β_2 and β_5 since there are no other axes in that plane. It was proved in Art. 35 that if a quadratic vector of constant plane has one axis in that plane a zero, and only one other axis in that plane, the plane itself is the tangent plane to the cones (3) at the zero. That is, the tangent plane to (3) at β_2 passes through β_5.

Consider the normal form (159). The tangent plane to the cones (3) at β_2 is the plane $(23\rho) = 0$. Hence we must have $(235) = 0$.

The form (159) now becomes, by putting (235) = 0,

$$\frac{\{\beta_4(527) + \beta_5(247)\}(52\rho)^2}{(527)^2} + \frac{\beta_2(23\rho)\,(45\rho)}{(237)} - \frac{\beta_5(524)\,(52\rho)\,(72\rho)}{(527)^2} \tag{207}$$

We note that the constant a cannot be zero, since this would give β_2 an inflectional element of the cones (3) and β_5 could not then lie in the tangent plane if $F\rho$ is irreducible. Also, (457) cannot vanish, since the planes $(45\rho) = 0$ and $(25\rho) = 0$ taken together constitute a quadric cone which would then contain six axes, as the limiting form of a reducible vector.

Let now the vector (207) be thrown into the form of the right member of (201) by addition of the term

$$\frac{-\rho(52\rho)\,(452)}{(527)}$$

which renders β_7 a zero without destroying the zero character of β_2 or β_5. By use of the identity

$$\rho(452) = \beta_4(52\rho) + \beta_5(24\rho) + \beta_2(45\rho)$$

we find the vector takes the form

$$\frac{\beta_5(52\rho)\{(247)\,(52\rho) - (524)\,(72\rho) - (24\rho)\,(527)\}}{(527)} + \beta_2\left\{\frac{(23\rho)\,(45\rho)}{(237)} - \frac{(45\rho)\,(52\rho)}{(527)}\right\}$$

the terms in β_4 destroying each other. But we have identically

$$(247)\,(52\rho) - (527)\,(24\rho) = (452)\,(72\rho)$$

whence the coefficient of β_5 vanishes. The vector is thus reducible, contrary to hypothesis. Hence its plane cannot be constant.

Case 2. The three axes β_2, β_5, and β_7 are coplanar. The form (159) becomes, putting (257) = 0,

$$\frac{\{\beta_5(247) + \beta_2(457)\}\,(23\rho)\,(45\rho)}{(237)\,(457)} + (a_1\beta_5 + b_1\beta_2)\,(52\rho)^2, \tag{208}$$

where a_1 and b_1 are constants, which, like the original constants a and b, cannot vanish in this case, $F\rho$ being irreducible. It is not

possible to throw this vector into the form (201), because we have not three diplanar axes. The form (203) is still possible. By the identity

$$\beta_7(452) = \beta_5(247) + \beta_2(457)$$

the vector (208) may be written, (neglecting a multiplicative constant),

$$\beta_7(23\rho)\ (45\rho) + (a_1\beta_5 + b_1\beta_2)\ (52\rho)^2,$$

the constants a_1 and b_1 being arbitrary but not zero. By subtraction of the vector term $\rho\ (237)\ (45\rho)$ the axis β_7 is made a zero. We may also write, identically,

$$\rho(237) = \beta_2(37\rho) + \beta_3(72\rho) + \beta_7(23\rho),$$

which gives us our quadratic vector as

$$-\beta_2(37\rho)\ (45\rho) - \beta^3(72\rho)\ (45\rho) + (a_1\beta_5 + b_1\beta_2)\ (52\rho)^2 \qquad (209)$$

The axes β_2, β_5, and β_7 are coplanar, so that we may write

$$\beta_5 = m\beta_2 + n\beta_7.$$

We therefore have $(52\rho) = n(72\rho)$.

If we now put $x_1 = (37\rho)$, $z_2 = (45\rho)$, and $x_3 = (72\rho)$, we shall have thrown (209) into the form (203). It is evident that the vector coefficients are not coplanar and the factorization proceeds as in Art. 34.

All quadratic vectors, therefore, having a triple axis, but no axis of higher order, can be thrown into the form $V\phi\rho\theta\rho + \rho S\delta\rho$.

37. Taking, finally, vectors having axes of order higher than the third, it has already been shown that these differ in their properties according as the cones (3) have, all of them, a double element at the multiple axis; or have a uniquely determined tangent plane there.

If β_1 is an axis of the fourth or higher order, and if we are dealing with the case of vanishing polar vector, so that the tangent plane is not unique or determinate; and if, also, there exist two other axes diplanar with β_1, the form (201) is not possible. For we can throw at once into the form (165), obviously a vector of constant plane, incapable, therefore, of being factored as $V\phi\rho\theta\rho$.

If, however, we subtract from (165) the term $\rho A_{12}x_3$, the resulting quadratic vector may be written

$$A_{13}\beta_1 x_1 x_2 - A_{12}\beta_3 x_3^2 + (A_{11}\beta_1 + A_{21}\beta_2 - A_{12}\beta_2 + A_{31}\beta_3) x_2 x_3, \quad (210)$$

which is in the form (203) and can therefore be factored into $V\phi\rho\theta\rho$, as in Art. 34, provided the vector coefficients are not in the same plane. It was shown in Art. 27 that neither A_{13} nor A_{12} can be zero. The condition that the vector coefficients be coplanar is therefore

$$A_{21} - A_{12} = 0. \quad (211)$$

If this condition is satisfied, (210) cannot be factored into $V\phi\rho\theta\rho$. If, however, we subtract from (165) the term $\rho A_{13} x_2$, the resulting quadratic vector may be written

$$A_{12}\beta_1 x_3 x_1 - A_{13}\beta_2 x_2^2 + (A_{11}\beta_1 + A_{21}\beta_2 + A_{31}\beta_3 - A_{13}\beta_3) x_2 x_3, \quad (212)$$

which may be factored like (203) if the vector coefficients are diplanar, that is unless

$$A_{31} - A_{13} = 0. \quad (213)$$

Suppose (211) and (213) both satisfied. Subtract from (165) the term $\rho(A_{12}x_3 + A_{13}x_2)$. The resulting quadratic vector may be written

$$- A_{13}\beta_2 x_2^2 - A_{12}\beta_3 x_3^2 + A_{11}\beta_1 x_2 x_3. \quad (214)$$

Neither A_{12} nor A_{13} can be zero, the vector being irreducible by hypothesis. If A_{11} is not zero we may put

$$\left.\begin{aligned} \phi\rho &= A_{11}x_2 V\beta_1\beta_2 + A_{12}x_3 V\beta_2\beta_3 \\ \theta\rho &= \frac{A_{13}x_2}{A_{11}S\beta_1\beta_2\beta_3} V\beta_2\beta_3 + \frac{x_3}{S\beta_1\beta_2\beta_3} V\beta_3\beta_1 \end{aligned}\right\} \quad (215)$$

and we then have (214) identically equal to $V\phi\rho\theta\rho$. If, on the other hand, $A_{11} = 0$, (214) cannot be factored into $V\phi\rho\theta\rho$. The resulting simple vector

$$A_{13}\beta_2 x_2^2 + A_{12}\beta_3 x_3^2 \quad (216)$$

may, however, be factored by first adding a term $\rho S\delta\rho$, where δ may be chosen in an infinite number of ways. It can be shown that any term

$$- \rho(bx_2 + cx_3)$$

will suffice, if b and c satisfy

$$A_{12}A_{13} - bA_{12} - cA_{13} = 0.$$

A simple solution is found by multiplying (216) by 2, and writing $b = A_{13}$, and $c = A_{12}$. The resulting quadratic vector is

$$2(A_{13}\beta_2 x_2^2 + A_{12}\beta_3 x_3^2) - \rho(A_{13}x_2 + A_{12}x_3)$$

which by taking $\rho = x_1\beta_1 + x_2\beta_2 + x_3\beta_3$ becomes

$$\beta_1(-A_{13}x_1x_2 - A_{12}x_1x_3) + \beta_2(A_{13}x_2^2 - A_{12}x_2x_3) + \beta_3(A_{12}x_3^2 - A_{13}x_2x_3), \tag{217}$$

which, by Art. 4, has the same axes as (216). If we take

$$\left.\begin{aligned} \phi\rho &= x_2 V\beta_1\beta_2 + x_3 V\beta_3\beta_1, \\ \theta\rho &= (A_{12}x_3 - A_{13}x_2)V\beta_2\beta_3 + x_1(A_{12}V\beta_1\beta_2 - A_{13}V\beta_3\beta_1), \end{aligned}\right\} \tag{218}$$

we shall have (217) identically equal to $V\phi\rho\theta\rho$, aside from the scalar factor $S\beta_1\beta_2\beta_3$.

This completes the factorization of (165) for all possible cases.

It was shown in Art. (27) that when the quadratic vector has a triple axis, (besides the quadruple axis with vanishing polar vector), it can be thrown into the form (170), evidently of constant plane. If, however, we subtract the term

$$\rho(A_{12} - A_{21})x_3 + \rho A_{13}x_2$$

the resulting quadratic vector may be written

$$-x_2^2 A_{13}\beta_2 - x_2x_3(A_{12}\beta_2 - A_{21}\beta_2 + A_{13}\beta_3) + x_3^2(a_1\beta_1 + a_2\beta_2 - A_{12}\beta_3 + A_{21}\beta_3) \tag{219}$$

which is the same form as (214) and can be factored as in (215) provided the vector coefficients are diplanar. We cannot have $A_{13} = 0$. The condition of coplanarity for the three coefficients is therefore $a_1 = 0$; that is, the element β_2 is an inflectional element of the cone (3). If such is the case, we may, instead, subtract from (170) the term $\rho(A_{12} - A_{21})x_3$. The resulting quadratic vector may be written

$$x_1x_2A_{13}\beta_1 + x_3^2(a_1\beta_1 + a_2\beta_2 - A_{12}\beta_3 + A_{21}\beta_3) - x_2x_3(A_{12} - A_{21})\beta_2 \tag{220}$$

which is in the form (203) and can be factored into $V\phi\rho\theta\rho$ if the vector coefficients are diplanar; that is, unless

$$A_{12} - A_{21} = 0.$$

If such is the case, and if, also, $a_1 = 0$, the vector (170) takes the simple form

$$\beta_1 A_{13} x_1 x_2 + \beta_2 a_2 x_3^2 \tag{221}$$

If we subtract the term $\rho x_3 \sqrt{a_2 A_{13}}$, the resulting quadratic vector may be written

$$\beta_1(A_{13}x_1x_2 - bx_2x_3) + \beta_2(a_2x_3^2 - bx_2x_3) - \beta_3 b x_3^2, \tag{222}$$

where $b = \sqrt{a_2 A_{13}}$. If we take

$$\begin{aligned} \phi\rho &= \sqrt{a_2}\, x_3 V\beta_2\beta_3 + \sqrt{A_{13}}\, x_1 V\beta_3\beta_1, \\ \theta\rho &= (\sqrt{a_2}\, x_3 - \sqrt{A_{13}}\, x_2) V\beta_1\beta_2 + \sqrt{A_{13}}\, x_3 V\beta_3\beta_1, \end{aligned} \tag{223}$$

we shall have (222) identically equal to $V\phi\rho\theta\rho$ aside from the scalar factor $S\beta_1\beta_2\beta_3$.

We have thus considered all possibilities for an axis β_1 when the cones (3) all have a double element at β_1, and β_1 is not of higher than the fourth order. If β_1 is of fifth or higher order we may assume β_2 and β_3 any two vectors not coplanar with β_1. By virtue of the vanishing polar vector we may then throw the quadratic vector into the form

$$a_1 x_2^2 + a_2 x_2 x_3 + a_3 x_3^2, \tag{224}$$

and if the vectors a are diplanar we may factor as in (214) and (215). If the a's are coplanar, β_1 must lie in their plane, for if not we shall have three other axes in that plane, (not necessarily distinct), and β_1 will be of the fourth order only. We may suppose β_2, as yet arbitrary, to be some other vector in this plane. (224) may then be written

$$\beta_1(B_{11}x_2^2 + B_{12}x_2x_3 + B_{13}x_3^2) + \beta_2(B_{21}x_2^2 + B_{22}x_2x_3 + B_{23}x_3^2), \tag{225}$$

where the B's are constant scalars. The plane of β_1 and β_2 is the same as the plane $x_3 = 0$. It is evident, therefore, that the only axis, if any, which (225) possesses distinct from β_1 is

$$\beta_1 B_{11} + \beta_2 B_{21}. \tag{226}$$

If B_{21} is not zero we may suppose β_2, as yet arbitrary in the plane $x_3 = 0$, to be an axis; when B_{11} will disappear. By subtracting the term $\rho B_{21} x_2$, the resulting quadratic vector may be written

$$-x_1x_2B_{21}\beta_1+x_2x_3(B_{12}\beta_1+B_{22}\beta_2-B_{21}\beta_3)+x_3^2(B_{13}\beta_1+B_{23}\beta_2), \quad (227)$$

the axis β_2 being made a zero, and, consequently, the term in x_2^2 disappearing. (227) is in the form (203) and can be factored into $V\phi\rho\theta\rho$ if the vector coefficients are diplanar. We may not have $B_{21} = 0$, for if so the vector is reducible. Thus the condition for coplanarity is that B_{23} shall vanish. If so, the quadratic vector (225) becomes

$$\beta_1(B_{12}x_2x_3 + B_{13}x_3^2) + \beta_2(B_{21}x_2^2 + B_{22}x_2x_3); \quad (228)$$

and if β_{12} is not zero we may subtract the vector term

$$\rho\left\{B_{21}x_2 + \frac{B_{21}B_{13}}{B_{12}}x_3\right\} \quad (229)$$

writing the resulting quadratic vector as

$$\beta_1(B_{12}x_2x_3+B_{13}x_3^2-B_{21}x_1x_2-bx_1x_3)+\beta_2(B_{22}-b)x_2x_3-\beta_3(B_{21}x_2x_3 +bx_3^2) \quad (230)$$

where b stands for the coefficient of x_3 in (229). Taking
$$\left.\begin{aligned}\phi\rho &= x_2V\beta_1\beta_2(B_{21}B_{13}-B_{22}B_{12})-V\beta_3\beta_1(x_3B_{21}B_{13}+x_2B_{12}B_{21})\\ \theta\rho &= B_{21}x_3V\beta_2\beta_3+(B_{12}x_3-B_{21}x_1)V\beta_1\beta_2,\end{aligned}\right\} \quad (231)$$

we have (230) identically equal to $V\phi\rho\theta\rho$ aside from the scalar factor $B_{21}B_{12}\cdot S\beta_1\beta_2\beta_3$.

If $B_{12} = 0$ (225) becomes

$$\beta_1B_{13}x_3^2 + \beta_2(B_{21}x_2^2 + B_{22}x_2x_3); \quad (232)$$

if we add the term $-c\rho(x_2B_{21} + x_3B_{22})$, where c is a scalar constant neither zero nor unity, the resulting quadratic vector may be written

$$\beta_1(B_{13}x_3^2-cB_{21}x_1x_2-cB_{22}x_1x_3)+\beta_2(1-c)(B_{21}x_2^2+B_{22}x_2x_3)-c\beta_3(B_{21}x_2x_3 +B_{22}x_3^2) \quad (233)$$

If we take

$$\begin{aligned}\phi\rho &= \frac{c^2B_{21}x_1}{1-c}V\beta_3\beta_1+c((x_2B_{21}+x_3B_{22})V\beta_2\beta_3+(x_3B_{13}-cx_1B_{22})V\beta_1\beta_2,\\ \theta\rho &= \frac{1-c}{c}x_2V\beta_1\beta_2 + x_3V\beta_3\beta_1,\end{aligned} \quad (234)$$

we shall have (233) identically equal to $V\phi\rho\theta\rho$ aside from the scalar factor $S\beta_1\beta_2\beta_3$.

Returning to (225), if $B_{21} = 0$, the vector can have no axis except β_1. If we take

$$y = -\frac{B_{22}}{c}, z = -\frac{B_{23}}{c}, \tag{235}$$

where c is any non-vanishing scalar, and subtract from (225) the term $\rho(yx_2 + zx_3)$, the resulting quadratic vector may be written

$$\beta_1(B_{11}x_2^2+B_{12}x_2x_3+B_{13}x_3^2-yx_1x_2-zx_1x_3)+\beta_2(B_{22}x_2x_3-zx_2x_3-yx_2^2) \\ -\beta_3(yx_2x_3+zx_3^2). \tag{236}$$

If we now take

$$\left.\begin{aligned}\phi\rho &= (x_2B_{22}+x_3B_{23})V\beta_2\beta_3-(x_1B_{22}+x_2B_{11}c)V\beta_3\beta_1 \\ &\quad +[x_1(cB_{22}-B_{23})+x_2(c^2B_{11}-cB_{12})-cB_{13}x_3]V\beta_1\beta_2, \\ \theta\rho &= -x_3V\beta_3\beta_1+(x_2+cx_3)V\beta_1\beta_2\end{aligned}\right\} \tag{237}$$

we shall have (236) identically equal to $V\phi\rho\theta\rho$ aside from the scalar factor $cS\beta_1\beta_2\beta_3$.

This completes the factorization into $V\phi\rho\theta\rho$ for all possible cases where all cones (3) have β_1 a double element.

If the cones (3) have β_1 an axis of the fourth or higher order, with a uniquely determined tangent plane, it has been shown that we may throw into one of the two forms (185) or (195), according as β_1 is an ordinary, or an inflectional, element.

The sub-case (185) may be at once factored if the vectors π, μ, and $\beta_1 u$ are diplanar. For if we take

$$\left.\begin{aligned}\phi\rho &= x_2V\mu\pi - ux_3V\pi\beta_1, \\ \theta\rho &= x_1V\pi\beta_1 - x_2V\beta_1\mu + x_3\left(\frac{g}{u}V\mu\pi + g_1V\pi\beta_1\right)\end{aligned}\right\} \tag{238}$$

we shall have (185) identically equal to $V\phi\rho\theta\rho$ aside from the factor $S\pi\mu\beta_1$.

If $u = 0$, this method fails. If there exists an axis not in the plane $x_2 = 0$ we may suppose, as already shown, that $\mu = a\beta_2$. Then the constant a cannot be zero. If we now subtract the term $a\rho x_2$, the resulting quadratic vector may be written

$$x_1x_2(\pi - a\beta_1) + x_2x_3(g_1\pi - a\beta_3) + x_3^2g\pi, \tag{239}$$

which factors like (203) if the vector coefficients are not in the same plane, that is unless $S\beta_3\beta_1\pi = 0$. If so, we must either have π parallel to β_1, or may take it parallel to β_3, which is as yet any vector in the tangent plane. If π is parallel to β_3 we may subtract the term $\rho(ax_2 + gx_3)$ from (186). The resulting quadratic vector may be written

$$x_2x_3\{(g_1 - a)\beta_3 - g\beta_2\} - x_3x_1g\beta_1 + x_1x_2\{c\beta_3 - a\beta_1\}, \quad (240)$$

where c is a constant which cannot vanish. This is in the form (201) and can be factored, since neither c nor g are zero for this case. If π is parallel to β_1, the tangent plane to (3) ceases to be uniquely determined, contrary to hypothesis.

If there exists no axis without the tangent plane, u being still zero, we must have $S\beta_1\pi\mu = 0$; for if not we should have three axis, (not necessarily distinct), aside from β_1. These cannot be distinct and in the tangent plane, nor, as was shown, can they be coincident in that plane. Hence $S\beta_1\pi\mu = 0$. But a quadratic vector of constant plane, having a zero in that plane, and only one other axis in that plane, must have that plane tangent to the cones (3) at the zero.[21] Hence π lies in the plane $x_2 = 0$, and may be taken to be parallel to β_3. Since μ is in the same plane we may take $\mu = b_1\beta_1 + b_3\beta_3$. The vector (185) now takes the form

$$\beta_3(ax_1x_2 + gx_3^2 + g_1x_2x_3) + (b_1\beta_1 + b_3\beta_3)x_2^2, \quad (241)$$

where a is a non-vanishing constant.

To see whether a term $\rho S\delta\rho$ can be added to this vector so that it can be factored into $V\phi\rho\theta\rho$, the most elegant method would be to consider, after Hamilton, the pure and the rotational parts of strains defined by ϕ and θ. As I have not in the present paper introduced these ideas, I shall employ the more cumbrous method of undetermined coefficients; and shall thereby avoid a digression upon simultaneous forms of ϕ and θ. Since the vectors β_1, β_2, and β_3 are diplanar, a general form of ϕ and θ with undetermined constants p and q may be taken as

$$\begin{aligned}\phi\rho = {} & x_1(pV\beta_2\beta_3 + qV\beta_3\beta_1 + rV\beta_1\beta_2) + x_2(p'V\beta_2\beta_3 + q_1V\beta_3\beta_1 + r'V\beta_1\beta_2) \\ & + x_3(p''V\beta_2\beta_3 + q''V\beta_3\beta_1 + r''V\beta_1\beta_2), \\ \theta\rho = {} & x_1(p_1V\beta_2\beta_3 + q_1V\beta_2\beta_1 + r_1V\beta_1\beta_2) + x_2(p'_1V\beta_2\beta_3 + q'_1V\beta_3\beta_1 \\ & + r'_1V\beta_1\beta_2) + x_3(p''_1V\beta_2\beta_3 + q''_1V\beta_3\beta_1 + r''_1V\beta_1\beta_2).\end{aligned}$$

21 Proved in Art. 35.

On the one hand we have the vector product of these two expressions. On the other, we have to add to (241) a term which we may write

$$-(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3)(x x_1 + y x_2 + z x_3),$$

where x, y, and z are undetermined, and are the components of the vector δ. Multiplying out, and equating coefficients of like terms in the variables x_1, x_2, x_3, we have a system of eighteen equations. It is clear at the start that we may take $x = 0$, for this is the same as saying that β_1 must be a zero; which follows from the facts: that β_1 is a sextuple axis; and $V\varphi\rho\theta\rho$ has 3 zeros. Also, we may take $p_1 = q_1 = r_1 = 0$. For this is the same as making the coefficient of x_1^2 zero, a necessary condition that β_1 be a zero. We then have the system of equations, fifteen in number,

Coef. of x_2^2.	Coef. of x_3^2.	Coef. of x_1x_2
1. $q'_1r' - q'r'_1 = b_1$	4. $p''_1q'' - p''q''_1 = g - z$	7. $q'_1r - qr'_1 = -y$
2. $p'r'_1 - p'_1r' = -y$	5. $p''r''_1 - p''_1r'' = 0$	8. $pr'_1 - p'_1r = 0$
3. $p'_1q' - p'q'_1 = b_3$	6. $q''_1r'' - q''r''_1 = 0$	9. $p'_1q - pq'_1 = a$
Coef. of x_2x_3		Coef. of x_3x_1
10. $q'_1r'' - q'r''_1 + q''_1r' - q''r'_1 = 0$		13. $q''_1r - q\,r''_1 = -z$
11. $p'r''_1 - p'_1r'' + p''r'_1 - p''_1r' = -z$		14. $pr''_1 - p''_1r = 0$
12. $p'_1q'' - p'q''_1 + p''_1q' - p''q'_1 = g_1 - y$		15. $p''_1q - p\,q''_1 = 0.$

Either $\phi\rho$ or $\theta\rho$ may be divided by a scalar which is also multiplied into the other, leaving $V\phi\rho\theta\rho$ unchanged. We may therefore without loss of generality assume that some one letter, as r'_1, is either zero or unity.

Case 1. Let $r'_1 = 1$. I shall show first that p must be zero. For, if not, we have, (by 8), $p = p'_1r$. Substituting in 14 we have, (since r cannot be zero if p is not zero), $p''_1 = p'_1r''_1$. Then writing for p and for p''_1 their values in 15 we have, since p'_1 cannot vanish if p does not,

$$r''_1q - q''_1r = 0. \qquad (A)$$

Comparing with 13, this gives $z = 0$. Hence r''_1 is not zero, for if so, by 13 and 14, (since r is not zero), $p''_1 = 0$ and $q''_1 = 0$, making $\theta\rho$ a monomial and $V\phi\rho\theta\rho$ reducible. We may then write $r''_1 = cr$, where c is not zero. By (A), $q''_1 = cq$ and by 14 $p''_1 = cp$. From 5 and 6 we have

$$p''r - pr'' = 0, \quad q''r - qr'' = 0,$$

and since r is not zero it follows that $p''q - pq'' = 0$. But from 4, writing for p''_1 and q''_1 their values, cp and cq,

$$-c(p''q - pq'') = g - z.$$

Therefore $g = z = 0$, contrary to fact, since if g vanishes the quadratic vector is reducible. Thus if $r'_1 = 1$, $p = 0$, which is equivalent to saying that either p or r'_1 must vanish.

Case 2. Let $p = 0$. We need not now assume $r'_1 = 1$. By 8, either p'_1 or r must vanish. Suppose $p'_1 = 0$. By 9, we have $a = 0$, But if $a = 0$ the vector (241) has, at β_1, a quadruple axis of the first kind, contrary to hypothesis.

Suppose $p = r = 0$. Then, by 9, neither p'_1 nor q can vanish. If we take $q = 1$, we have $p'_1 = a$, (by 9), and $r'_1 = y$, (by 7). Also $r''_1 = z$, (by 13), and $p''_1 = 0$, (by 15). Now r''_1 cannot vanish, for if so we have, by 6, either $q''_1 = 0$, or $r'' = 0$; but if $q''_1 = 0$, $g = 0$, by 4; and if $r'' = 0$, $p'' = 0$ by 11, (we cannot have $r'_1 = 0$, since not both y and z vanish), and again $g = 0$ (by 4). Hence r''_1 is not zero.

We therefore have $p'' = 0$, (by 5), and $z = g$, (by 4). Equation 6 now becomes

$$q_1''r'' - gq'' = 0, \tag{A}$$

which will serve to determine q''. There remain the six equations 1, 2, 3, 10, 11, 12. Substituting values, these become,

1. $q'_1r' - q'y = b_1$
2. $p'y - ar' = -y$
3. $aq' - p'q'_1 = b_3$
10. $q'_1r'' - q'g + q''_1r' - q''y = 0$
11. $p'g - ar'' = -g$
12. $aq'' - p'q''_1 = g_1 - y.$

By elimination of q'' and p' from 11, 12, and (A), we have

$$q''_1 = g_1 - y. \tag{B}$$

By elimination of q' and r' from 1, 2, and 3,

$$y(q'_1 - b_3) = ab_1. \tag{C}$$

Since neither a nor b_1 can be zero, y cannot. Hence this equation gives a value for q'_1. By 11, we have r'' in terms of p'. It is then an easy calculation to substitute, in 10, all other unknowns in terms of p' and y. We find all coefficients of powers of y cancel out, giving $b_1g = 0$, which is impossible since the vector is irreducible. Hence we cannot have $p = 0$.

Case 3. Let $r'_1 = 0$. We cannot also have $p'_1 = 0$; for if so, we have $y = 0$, (by 2), and $r = 0$, (by 7, since q'_1 is not zero by 9); then $r''_1 = 0$, (by 14), and $z = 0$, (by 13), but z and y are not both zero.

We then must have, by 8, $r'_1 = r = 0$. By 7, $y = 0$. By 2, $r' = 0$. By 1, $b_1 = 0$, which is not true. Hence r'_1 cannot be zero.

Thus the vector (241) cannot be written in the form

$$V\phi\rho\theta\rho + \rho S\delta\rho.$$

This completes the examination of (185) when $u = 0$. The factorization (238) also fails when $S\pi\mu\beta_1 = 0$. If there exists an axis not in the plane $x_2 = 0$, we may, as before, subtract the term $a\rho x_2$, with $\mu = a\beta_2$. If a does not vanish we may factor as in (203). We cannot now conclude that a is not zero, since we have not $u = 0$. If $a = 0$, and $S\pi\beta_1\beta_2 = 0$, the vector (185) becomes

$$(c_1\beta_1 + c_2\beta_2)\,(x_1x_2 + gx_3^2 + g'x_2x_3) + u\beta_1x_2x_3 \qquad (242)$$

where c_1 and c_2 are constants, and c_2, at least, is not zero since π is not parallel to β_1. If c_1 is not zero, we may add the term

$$-\rho\left(\frac{c_2u}{c_1}\right)$$

when the quadratic vector may be written

$$\begin{aligned}&\beta_1\{c_1x_1x_2 + c_1gx_3^2 + (c_1g_1 + u)x_2x_3 - zx_1x_3\}\\ &\quad + \beta_2\{c_2x_1x_2 + c_2gx_3^2 + c_2g_1x_2x_3 - zx_2x_3\} - \beta_3zx_3^2, \qquad (243)\end{aligned}$$

where z has been written for $\dfrac{c_2u}{c_1}$. If we then take

$$\begin{aligned}&\phi\rho = c_2x_3V\beta_2\beta_3 - c_1x_3V\beta_3\beta_1 - c_2x_1V\beta_1\beta_2,\\ &\theta\rho = V\beta_2\beta_3 + V\beta_3\beta_1\left\{\frac{ux_3}{c_1} - \frac{c_1x_2}{c_2}\right\} + V\beta_1\beta_2\left\{gx_3 + \left(g_1 + \frac{u}{c_1}\right)x_2\right\}, \qquad (244)\end{aligned}$$

we find (243) identically equal to $V\phi\rho\theta\rho$, aside from the factor $S\beta_1\beta_2\beta_3$.

This method fails if $c_1 = 0$. If so, the vector (242) becomes, aside from a scalar factor,

$$\beta_2(ax_1x_2 + gx_3^2 + g_1x_2x_3) + u\beta_1x_2x_3, \qquad (245)$$

where a is a constant which is not zero. This quadratic vector has β_1

for a quadruple axis and β_2 for a triple axis. If we add the vector term

$$-\rho(ax_1 + g_1x_3)$$

the resulting quadratic vector may be written

$$\beta_1(ux_2x_3 - ax_1^2) + (g\beta_2 - g_1\beta_3)x_3^2 + (-g_1\beta_1 - a\beta_3)x_3x_1, \qquad (246)$$

and if we then take

$$\begin{aligned}\phi\rho &= -ax_1(g_1V\beta_3\beta_1 + gV\beta_1\beta_2) - ax_2uV\beta_3\beta_1 \\ &\qquad + x_3(ga\,V\beta_2\beta_3 - g_1^2V\beta_3\beta_1 - gg_1V\beta_1\beta_2), \\ \theta\rho &= ax_1V\beta_3\beta_1 + x_3(g_1V\beta_3\beta_1 + gV\beta_1\beta_2), \qquad (247)\end{aligned}$$

we find (246) identically equal to $V\phi\rho\theta\rho$, aside from a scalar factor $gaS\beta_1\beta_2\beta_3$.

If no axis exists except in the plane $x_2 = 0$, (having now by hypothesis $S\pi\mu\beta_1 = 0$ and u not zero), we must, as before, have π an axis which we may take as β_3. The quadratic vector may then be written

$$\beta_3(ax_1x_2 + gx_3^2 + g_1x_2x_3) + u\beta_1x_2x_3 + (b_1\beta_1 + b_3\beta_3)x_2^2, \qquad (248)$$

where the constants a, b_1, and b_3, have the same meaning as in (241).

To see whether a term $\rho S\delta\rho$ can be found which shall render this vector factorizable, we may set up equations as for (241), with the difference that the right member of 10 will now be u instead of zero. In the equations under (241), for Cases 1 and 3, equation 10 was not used. Hence the reasoning still holds. For case 2, the reasoning is as before, up to the substitution in equation 10. The result gives $y = \dfrac{b_1g}{u}$ as a unique solution. The values of $\phi\rho$ and of $\theta\rho$ which follow are, on letting p', (which, from the equations, is arbitrary), have the value -1, and clearing of fractions,

$$\begin{aligned}\phi\rho &= -gx_2V\beta_2\beta_3 + (gx_1 - ux_2)V\beta_3\beta_1, \\ \theta\rho &= agux_2V\beta_2\beta_3 + \{(au^2 + b_3gu)x_2 + (gg_1u - b_1g^2)x_3\}V\beta_3\beta_1 \\ &\qquad + \{b_1g^2x_2 + g^2ux_3\}V\beta_1\beta_2;\end{aligned}$$

while (248), after subtracting the term $\rho(yx_2 + gx_3)$, becomes

$$\begin{aligned}\beta_1(ux_2x_3 + b_1x_2^2 - yx_1x_2 - gx_1x_2) + \beta_2(-yx_2^2 - gx_2x_3) + \beta_3(ax_1x_2 + g_1x_2x_3 \\ + b_3x_2^2 - yx_2x_3). \qquad (250)\end{aligned}$$

We then find that (250) is identically equal to $V\phi\rho\theta\rho$, aside from the scalar factor $g^2uS\beta_1\beta_2\beta_3$.

This completes all possible special forms of the vector (185). The only form which cannot be made factorizable by a term $\rho S\delta\rho$ is the vector (241).

The only remaining irreducible vector to be considered is (195). If there is at least one axis other than β_1, we can write as (196), which factors like (203) if the vector coefficients are diplanar, that is, unless $S\beta_3\beta_1\zeta = 0$. If so, we may write (196) as

$$\beta_1(b_1x_2x_3 + ux_3^2) + \beta_3(b_3x_2x_3 + ax_1x_2), \tag{251}$$

where $\zeta = b_1\beta_1 + b_3\beta_3$. To see whether a term $\rho S\delta\rho$ can be added to this vector so that it can be factored into $V\phi\rho\theta\rho$, we note that it has β_1 for a quintuple axis, and β_2 for a single axis, and the other axis is

$$\beta_1b_3b_1 + \beta_2au - \beta_3ab_1.$$

Therefore $V\phi\rho\theta\rho$, if it exists, must have β_1 for a zero, since three of its axes must be zeros, (not necessarily distinct). We may then set up equations, fifteen in number, as for the vector (241) and the left members will be identical with those for (241). The right members will be the same for equations 2, 5, 7, 8, 9, 11, 13, 14, and 15. The remaining equations become

1. $q'_1r' - q'r'_1 = 0$
3. $p'_1q' - p'q'_1 = 0$
4. $p''_1q'' - p''q''_1 = -z$
6. $q''_1r'' - q''r''_1 = u$
10. $q'_1r'' - q'r''_1 + q''_1r' - q''r'_1 = b_1$
12. $p'_1q'' - p'q''_1 + p''_1q' - p''q'_1 = b_3 - y$

The reasoning under case 1 is precisely as before, up to the obtaining of the relations $q''_1 = cq$ and $p''_1 = cp$, since none of the new equations are used so far. Then by 4 and 5 we have

$$p''r - pr'' = 0, \quad p''q - pq'' = 0,$$

and since p is not zero it follows that $q''r - qr'' = 0$. But, (from 6), putting for q''_1 and r''_1 their values cq and cr, we have

$$-c(q''r - qr'') = u$$

Therefore $u = 0$, which makes (251) reducible contrary to hypothesis. Hence if $r'_1 = 1$ we cannot have p different from zero. Hence either r'_1 or p must be zero.

Case 2. Let $p = 0$. As before, p'_1 cannot be zero. Hence $r = 0$. And as before, if $q = 1$ $p'_1 = a$, $r'_1 = y$, $r''_1 = z$, and $p''_1 = 0$.

We cannot have $r''_1 = 0$. For if so we cannot also have $q''_1 = 0$, for $\theta\rho$ would be a monomial. Then $p'' = 0$, (by 4), and $r'' = 0$, (by 11). This makes $u = 0$, (by 6), which is contrary to hypothesis.

We must then have $p'' = 0$,(by 5) and again $z = 0$, (by 4), or $r''_1 = 0$, just proved impossible. Hence this case is impossible.

Case 3. Let $r'_1 = 0$. We cannot have $p'_1 = 0$, by the same reasoning as before. We must then have $r = 0$, (by 8), $y = 0$, (by 7), and $r' = 0$, (by 2), as before. Then $r''_1 = 0$, (by 14), and $z = 0$, (by 13). But not both y and z are zero for this leaves (251) unchanged. Hence this case is impossible. Hence no term $\rho S\delta\rho$ can be found.

It remains to consider the possibility that (195) shall not possess an axis other than β_1. If we write $\zeta = b_1\beta_1 + b_2\beta_2 + b_3\beta_3$, and $u = c_1\beta_1 + c_2\beta_2 + c_3\beta_3$, the vector (195) becomes

$$\beta_1(b_1x_2x_3+c_1x_2^2+ux_3^2)+\beta_2(b_2x_2x_3+c_2x_2^2)+\beta_3(b_3x_2x_3+c_3x_2^2+ax_1x_2), \tag{252}$$

which we may denote as usual by $F\rho$. If $\rho = \beta_1x_1 + \beta_2x_2 + \beta_3x_3$, the vector equation $V\rho F\rho = 0$ defines the axes of $F\rho$, and, by multiplying out, is equivalent to the three scalar equations

$$\begin{aligned} x_2(b_3x_2x_3 + c_3x_2^2 + ax_1x_2) - x_3(b_2x_2x_3 + cx_2^2) &= 0, \\ x_3(b_1x_2x_3 + c_1x_2^2 + ux_3^2) - x_1(b_3x_2x_3 + c_3x_2^2 + ax_1x_2) &= 0, \\ x_1(b_2x_2x_3 + c_2x_2^2) - x_2(b_1x_2x_3 + c_1x_2^2 + ux_3^2) &= 0. \end{aligned}$$

If there is no axis except β_1, these equations have no solution when x_2 is not zero. Furthermore, if x_2 is not zero, any solution of the first and third equations simultaneously must be a solution of the second. By elimination of x_1 from the first and third equations we have the cubic

$$\begin{aligned} b_2^2x_3^3+(2b_2c_2-b_2b_3-au)x_2x_3^2+(c_2^2-b_3c_2-b_2c_3-ab_1)x_2^2x_3 \\ -(ac_1+c_2c_3)x_2^3=0. \end{aligned} \tag{253}$$

If this equation can be solved for x_3, we can find x_1 from the first of the above cubics, and so have an axis other than β_1. That no such axis exist it is necessary that $b_2 = 0$, and hence also that $au = 0$ contrary to hypothesis. Hence such an axis must exist. This completes the study of (195), and hence of irreducible quadratic vectors.

Summary of Results.

38. All irreducible quadratic vectors may be thrown into the form $V\phi\rho\theta\rho + \rho S\delta\rho$, with exception of two special cases.

First exception.

$$\beta_3(ax_1x_2 + gx_3^2 + g_1x_2x_3) + (b_1\beta_1 + b_3\beta_3)x_2^2. \qquad (241)$$

This vector may be detected, when its components X, Y, Z, are given in any form, by the following properties; it has a sextuple axis, and a single axis; the tangent plane to the cones (3) is unique and determinate for at least one of the cones; the single axis lies in this tangent plane; on taking β_1 for the multiple axis, β_3 for the single axis, (so that $x_2 = 0$ gives the tangent plane), and β_2 any vector without the tangent plane, we may remove the terms in x_1^2, x_1x_2, and x_1x_3 from the coefficient of β_1, by adding a properly chosen term $\rho S\delta\rho$; and the vector then takes the form (241).

Second exception.

$$\beta_1(b_1x_2x_3 + ux_3) + \beta_3(b_3x_2x_3 + ax_1x_2). \qquad (251)$$

The properties of this vector are: it has a quintuple axis; the tangent plane to at least one of the cones (3) is unique and determinate at this axis, which is an inflectional element of the cone; on taking β_1 for the quintuple axis, β_2 for another axis, β_3 any vector in the tangent plane except β_1, we may remove the terms x_1^2, x_1x_2, and x_1x_3 from the coefficient of β_1; the vector then takes the form (251).

If the quadratic vector does not come under either exceptional case, the method of throwing it into the form $V\phi\rho\theta\rho + \rho S\delta\rho$ depends upon the configuration of the axes. The form of the constant vector δ is not, usually, unique, but may have thirty-five possibilities or fewer. The values obtained in this paper, are, therefore, not the only ones that could be given for the majority of cases. They have been selected so that δ should be rational, and as simple as possible in terms of known axes.

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